CS 110
Computer Architecture
Lecture 19:
Amdahl's Law, Data-level Parallelism

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https://robotics.shanghaitech.edu.cn/courses/ca/21s/

School of Information Science and Technology SIST

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Slides based on UC Berkeley's CS61C

New-School Machine Structures (It's a bit more complicated!)

Software | *Hardware*

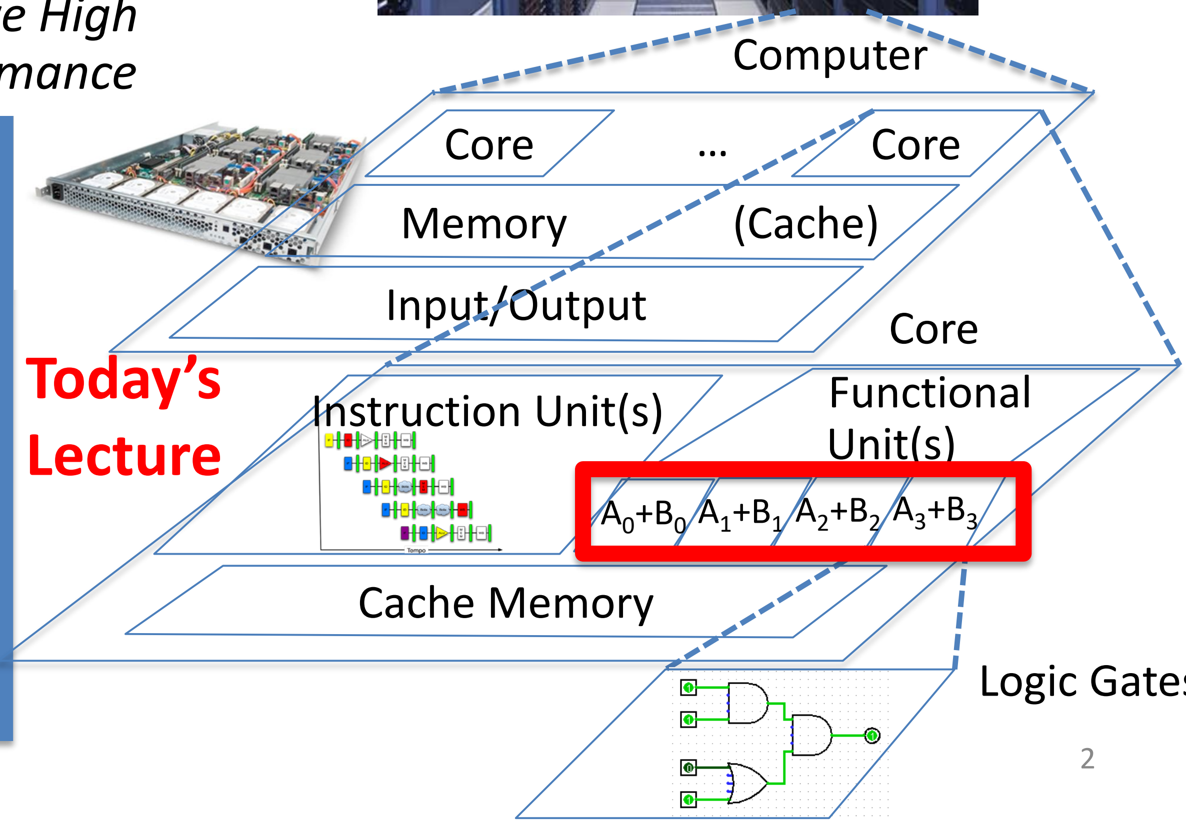
- Parallel Requests
  Assigned to computer
  e.g., Search "Katz"
- Parallel Threads
  Assigned to core
  e.g., Lookup, Ads
- Parallel Instructions
  >1 instruction @ one time
  e.g., 5 pipelined instructions
- Parallel Data
  >1 data item @ one time
  e.g., Add of 4 pairs of words
- Hardware descriptions
  All gates @ one time
- Programming Languages

Harness Parallelism & Achieve High Performance

Today's Lecture

Warehouse Scale Computer

Smart Phone



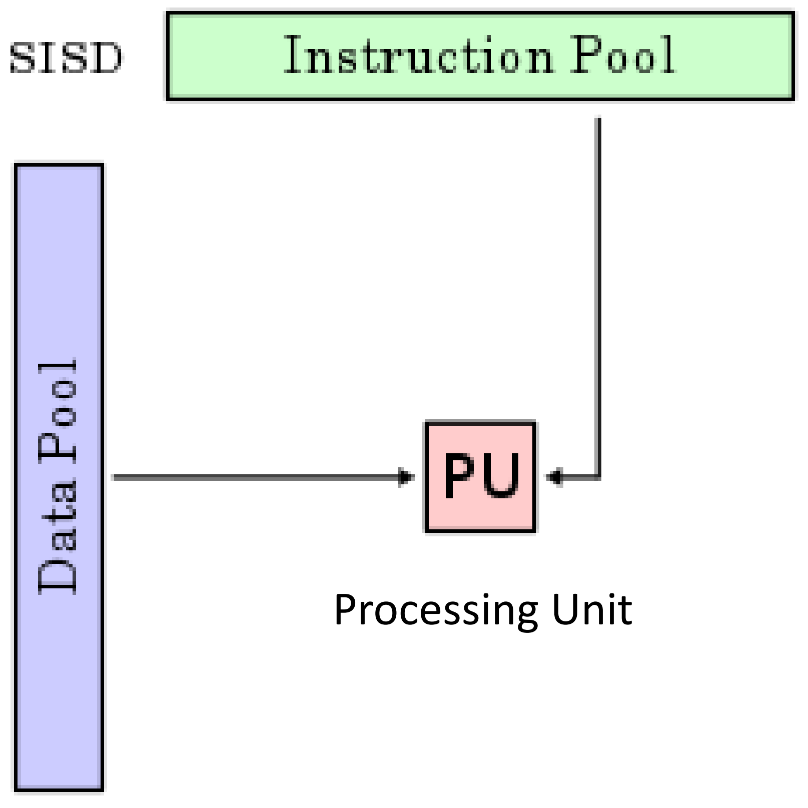
Why Parallel Processing?

- CPU Clock Rates are no longer increasing
  - Technical & economic challenges
    - Advanced cooling technology too expensive or impractical for most applications
    - Energy costs are prohibitive
- Parallel processing is only path to higher speed

Using Parallelism for Performance

- Two basic ways:
  - Multiprogramming
    - run multiple independent programs in parallel
    - “Easy”
  - Parallel computing
    - run one program faster
    - “Hard”
- We’ll focus on parallel computing for next few lectures

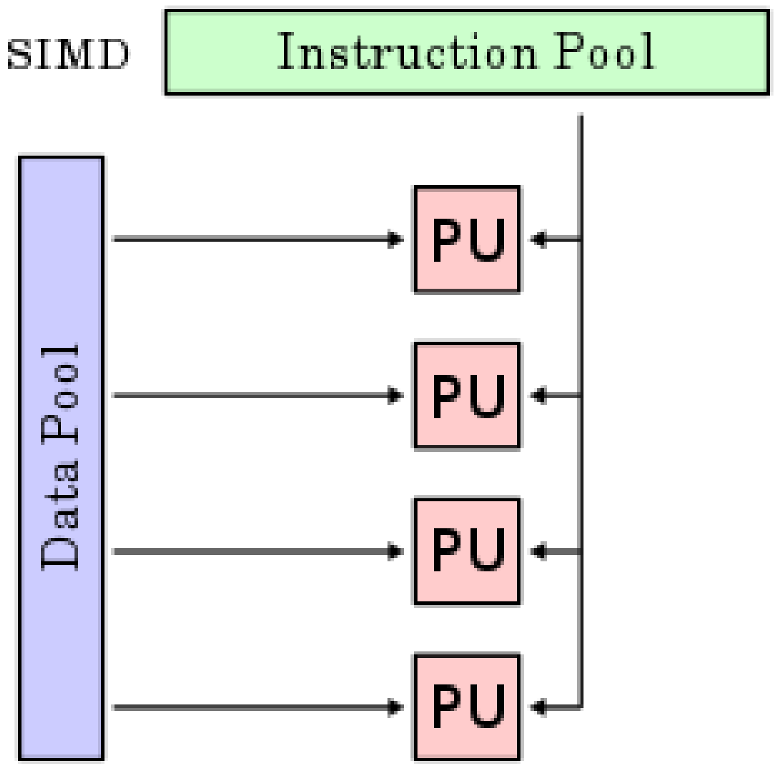
Single-Instruction/Single-Data Stream (SISD)



- Sequential computer that exploits no parallelism in either the instruction or data streams. Examples of SISD architecture are traditional uniprocessor machines
  - E.g. Our RISC-V processor
  - Superscalar is SISD because **programming model** is sequential

This is what we did up to now in CA.

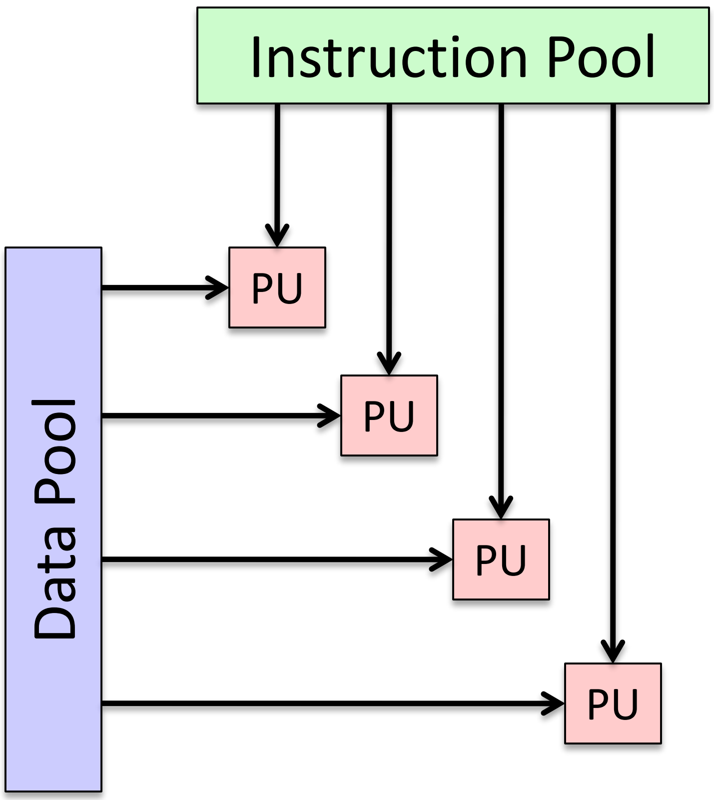
Single-Instruction/Multiple-Data Stream (SIMD or “sim-dee”)



- SIMD computer exploits multiple data streams against a single instruction stream to operations that may be naturally parallelized, e.g., Intel SIMD instruction extensions or NVIDIA Graphics Processing Unit (GPU)

Today’s topic.

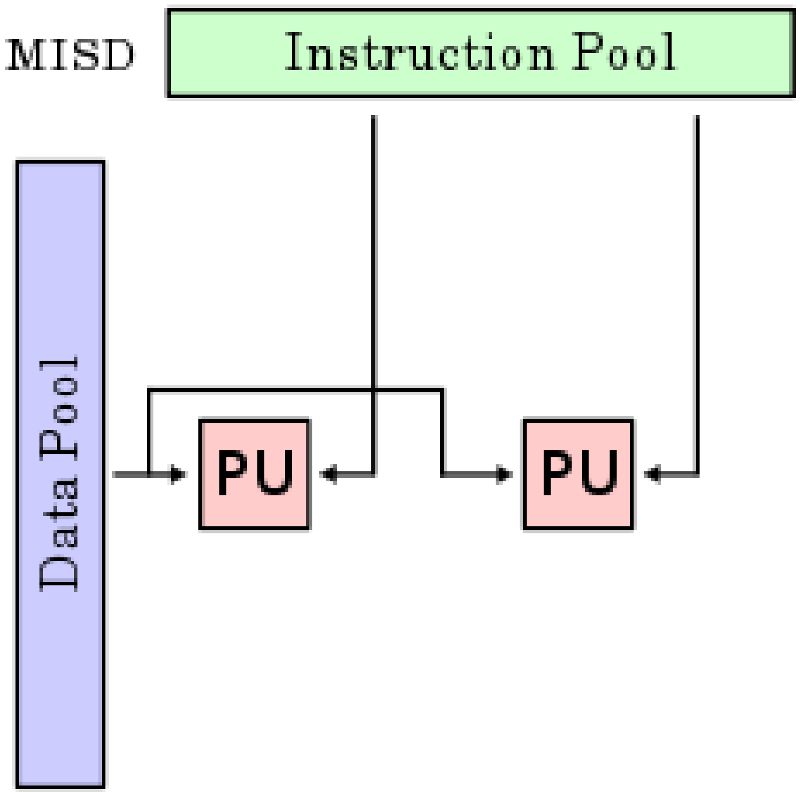
Multiple-Instruction/Multiple-Data Streams (MIMD or "mim-dee")



- Multiple autonomous processors simultaneously executing different instructions on different data.
  - MIMD architectures include multicore and Warehouse-Scale Computers

Next lecture & following.

Multiple-Instruction/Single-Data Stream (MISD)



- Multiple-Instruction, Single-Data stream computer that exploits multiple instruction streams against a single data stream.
  - Rare, mainly of historical interest only

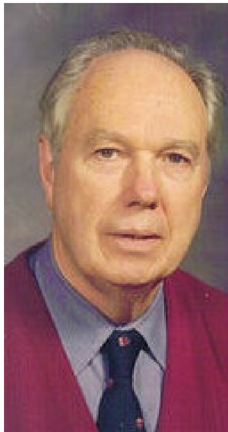
Few applications. Not covered in CA.

Flynn* Taxonomy, 1966

		Data Streams	
		Single	**Multiple**
Instruction Streams	Single	SISD: Intel Pentium 4	SIMD: SSE instructions of x86
	Multiple	MISD: No examples today	MIMD: Intel Xeon e5345 (Clovertown)

- Since about 2013, SIMD and MIMD most common parallelism in architectures – usually both in same system!
- Most common parallel processing programming style: Single Program Multiple Data ("SPMD")
  - Single program that runs on all processors of a MIMD
  - Cross-processor execution coordination using synchronization primitives
- SIMD (aka hw-level *data parallelism*): specialized function units, for handling lock-step calculations involving arrays
  - Scientific computing, signal processing, multimedia (audio/video processing)

*Prof. Michael Flynn, Stanford

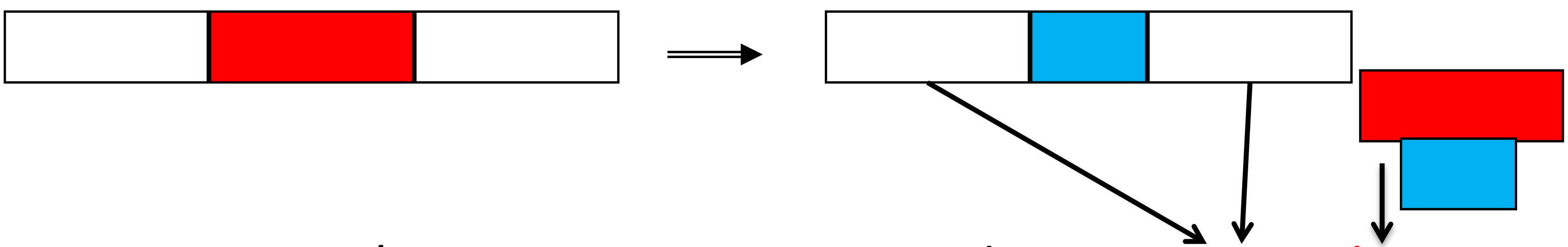


Big Idea: Amdahl's (Heartbreaking) Law

- Speedup due to enhancement E is

$$\text{Speedup w/ E} = \frac{\text{Exec time w/o E}}{\text{Exec time w/ E}}$$

- Suppose that enhancement E accelerates a fraction F (F <1) of the task by a factor S (S>1) and the remainder of the task is unaffected



Execution Time w/ E = Execution Time w/o E x [(1-F) + F/S]

Speedup w/ E = 1 / [(1-F) + F/S]

Big Idea: Amdahl's Law

$$\text{Speedup} = \frac{1}{(1 - F) + \frac{F}{S}}$$

Non-speed-up part → $(1 - F)$

Speed-up part → $\frac{F}{S}$

Example: the execution time of half of the program can be accelerated by a factor of 2.
What is the program speed-up overall?

$$\frac{1}{0.5 + \frac{0.5}{2}} = \frac{1}{0.5 + 0.25} = 1.33$$

Example #1: Amdahl's Law

$$\text{Speedup w/ E} = 1 / [(1-F) + F/S]$$

- Consider an enhancement which runs 20 times faster but which is only usable 25% of the time

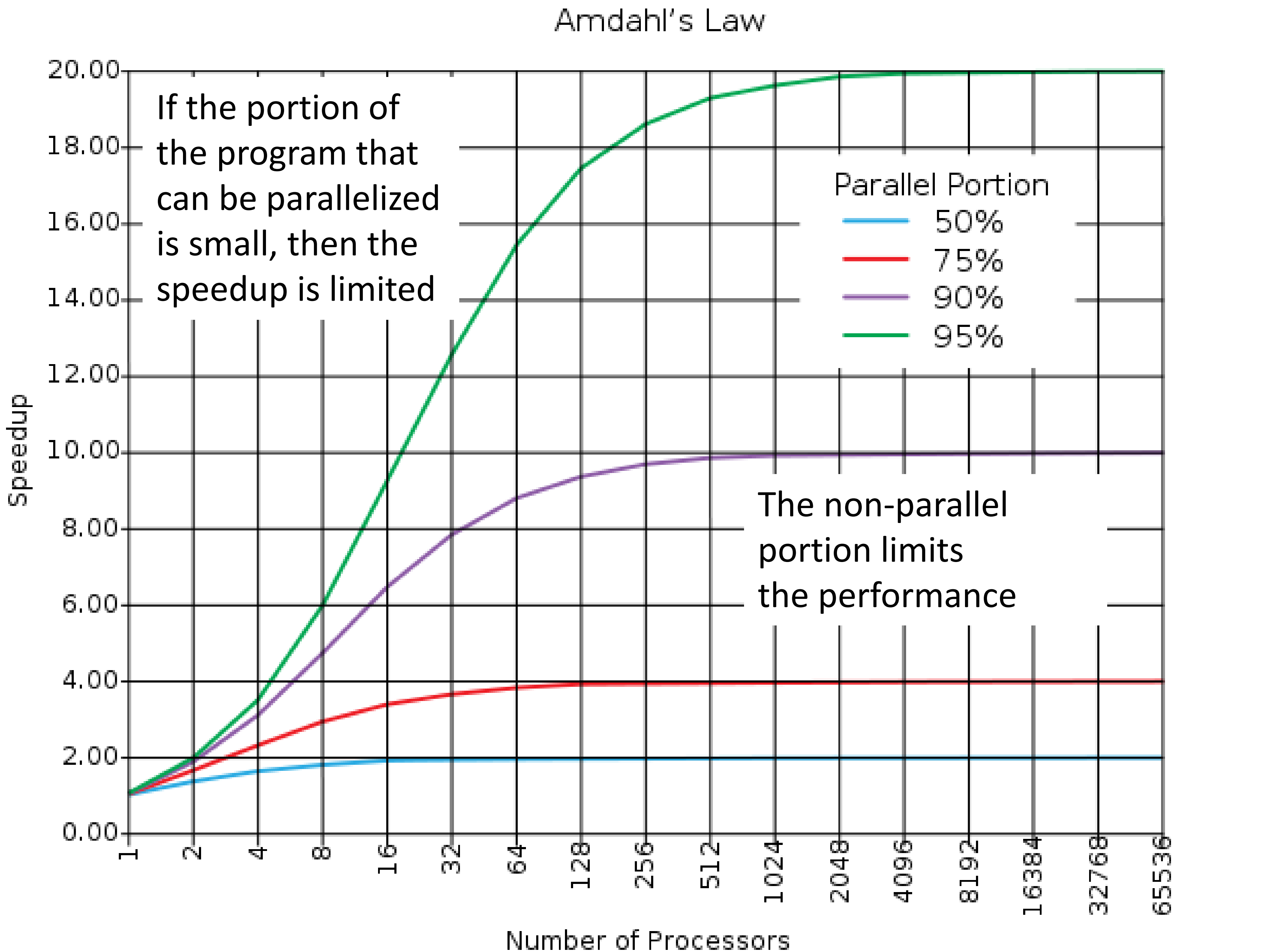
  $$\text{Speedup w/ E} = 1/(.75 + .25/20) = 1.31$$

- What if its usable only 15% of the time?

  $$\text{Speedup w/ E} = 1/(.85 + .15/20) = 1.17$$

- Amdahl's Law tells us that to achieve linear speedup with 100 processors, none of the original computation can be scalar!
- To get a speedup of 90 from 100 processors, the percentage of the original program that could be scalar would have to be 0.1% or less

  $$\text{Speedup w/ E} = 1/(.001 + .999/100) = 90.99$$



Amdahl's Law
If the portion of the program that can be parallelized is small, then the speedup is limited
Parallel Portion
50%
75%
90%
95%
The non-parallel portion limits the performance
Speedup
20.00
18.00
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
1
2
4
8
16
32
64
128
256
512
1024
2048
4096
8192
16384
32768
65536
Number of Processors

Strong and Weak Scaling

- To get good speedup on a parallel processor while keeping the problem size fixed is harder than getting good speedup by increasing the size of the problem.
  - *Strong scaling*: when speedup can be achieved on a parallel processor without increasing the size of the problem
  - *Weak scaling*: when speedup is achieved on a parallel processor by increasing the size of the problem proportionally to the increase in the number of processors
- Load balancing is another important factor: every processor doing same amount of work
  - Just one unit with twice the load of others cuts speedup almost in half

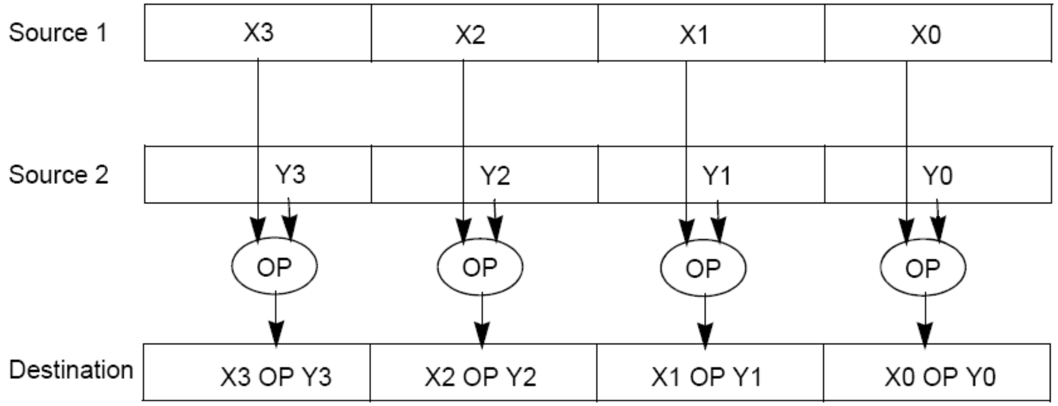
SIMD Architectures

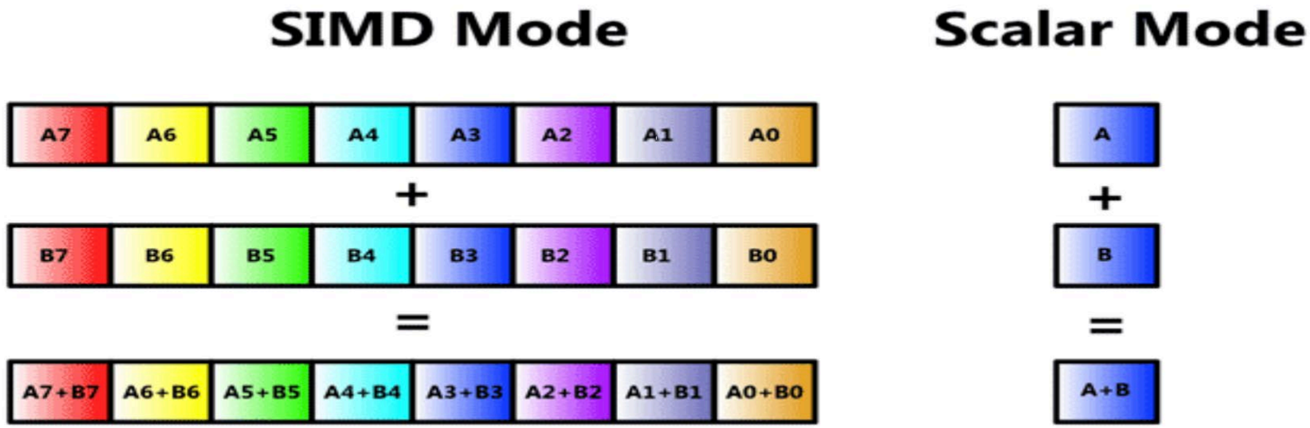
- *Data parallelism*: executing same operation on multiple data streams
- Example to provide context:
  - Multiplying a coefficient vector by a data vector (e.g., in filtering)

    $y[i] := c[i] \times x[i], \ 0 \leq i < n$
- Sources of performance improvement:
  - One instruction is fetched & decoded for entire operation
  - Multiplications are known to be independent
  - Pipelining/ concurrency in memory access as well
  - Special functional units may be faster

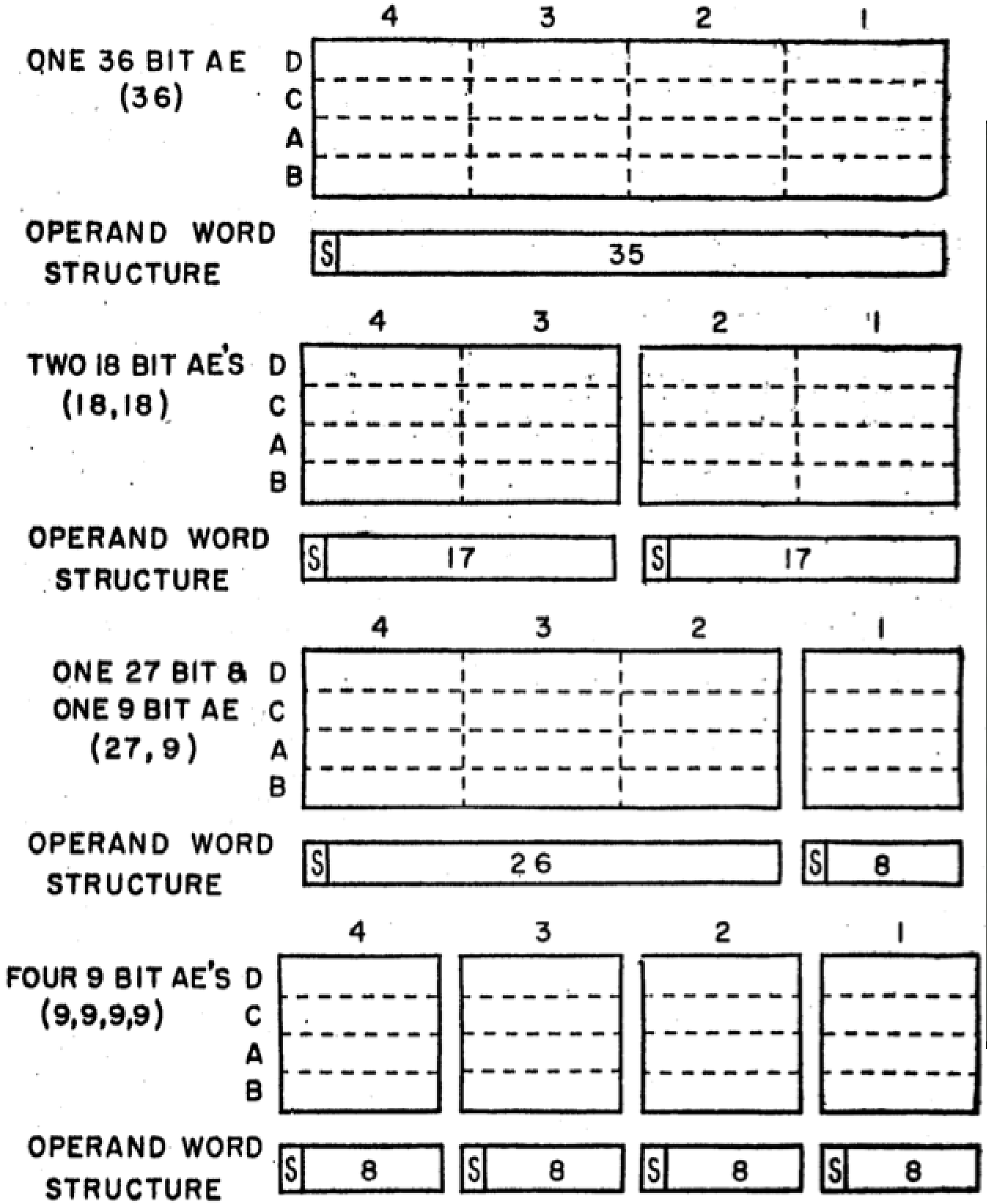
Intel "Advanced Digital Media Boost"

- To improve performance, Intel's SIMD instructions
  - Fetch one instruction, do the work of multiple instructions





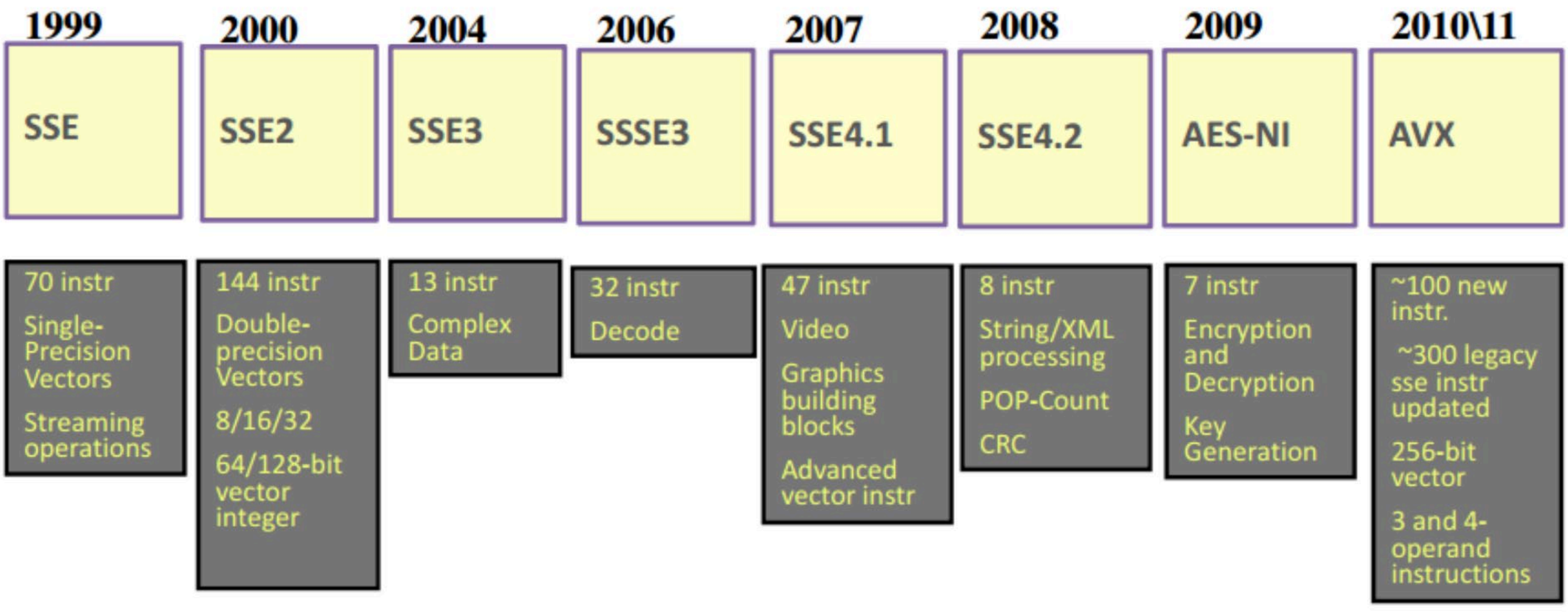
First SIMD Extensions: MIT Lincoln Labs TX-2, 1957



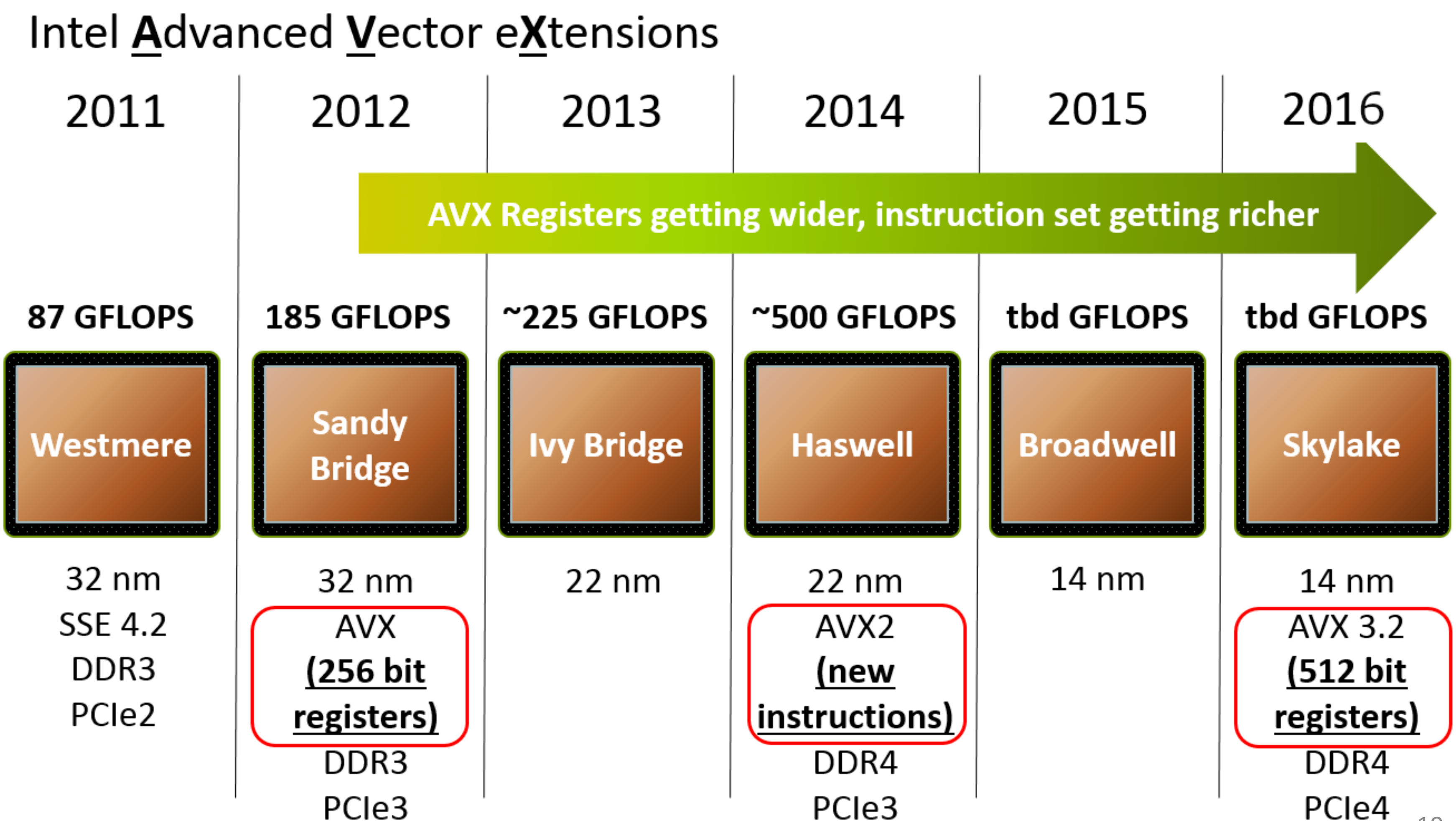
Intel SIMD Extensions

- MMX 64-bit registers, reusing floating-point registers [1992]

MMX 1997



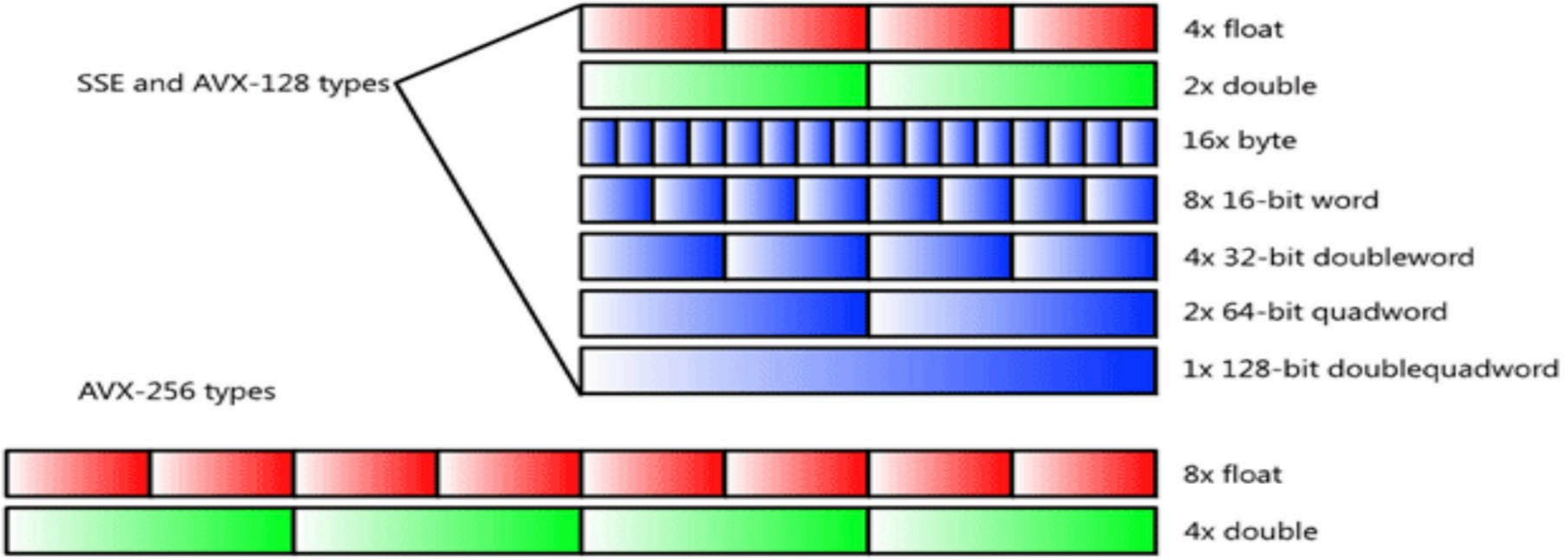
Intel Advanced Vector eXtensions AVX



https://chrisadkin.io/2015/06/04/under-the-hood-of-the-batch-engine-simd-with-sql-server-2016-ctp/

Intel Architecture SSE SIMD Data Types

- Note: in Intel Architecture (unlike RISC-V) a word is 16 bits
  - Single-precision FP: Double word (32 bits)
  - Double-precision FP: Quad word (64 bits)
  - AVX-512 available (16x float and 8x double)

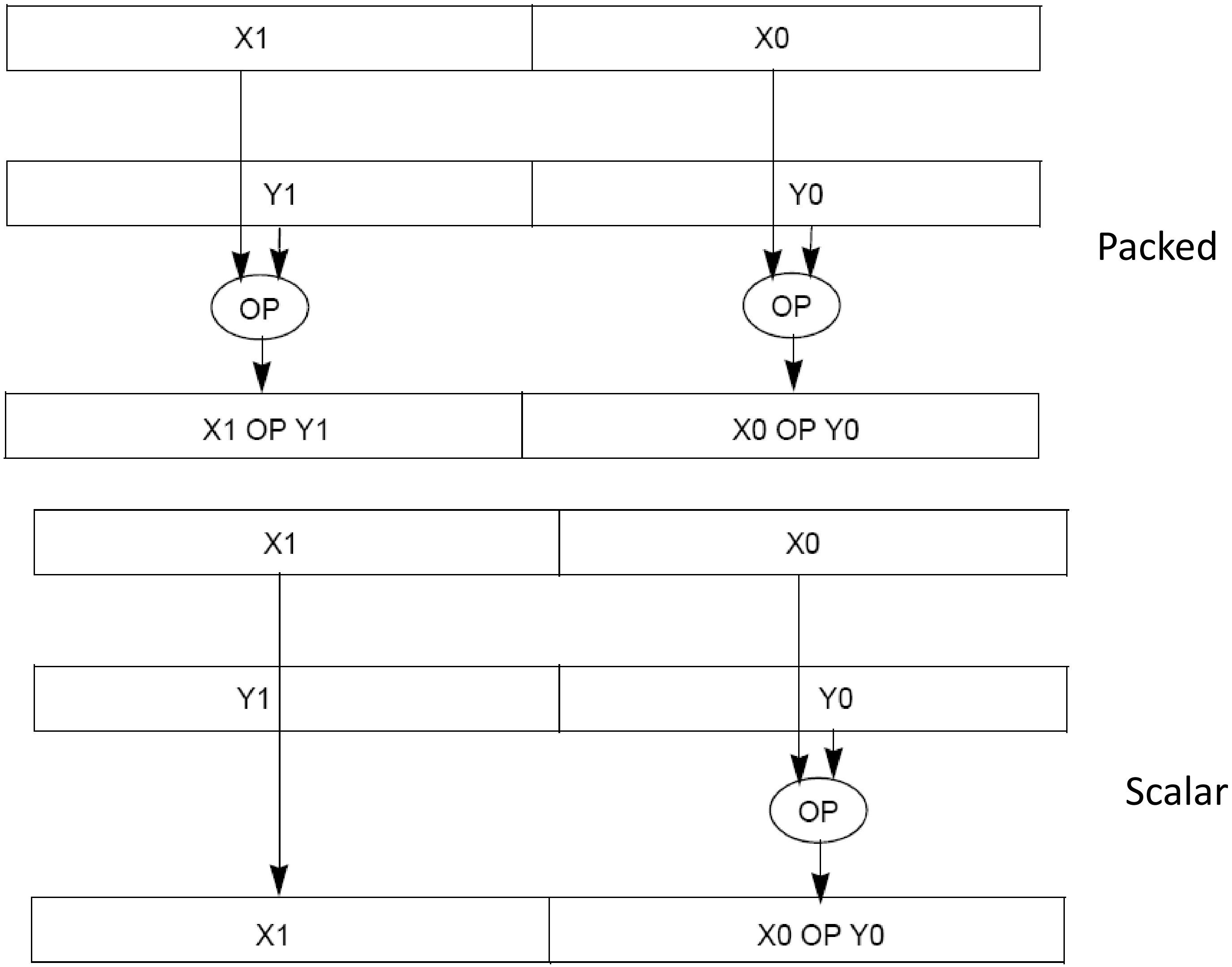


SSE/SSE2 Floating Point Instructions

	Data transfer	Arithmetic	Compare
Move does both load and store	MOV{A/U}{SS/PS/SD/PD} xmm, mem/xmm	ADD{SS/PS/SD/PD} xmm, mem/xmm	CMP{SS/PS/SD/PD}
		SUB{SS/PS/SD/PD} xmm, mem/xmm	
	MOV {H/L} {PS/PD} xmm, mem/xmm	MUL{SS/PS/SD/PD} xmm, mem/xmm	
		DIV{SS/PS/SD/PD} xmm, mem/xmm	
		SQRT{SS/PS/SD/PD} mem/xmm	
		MAX {SS/PS/SD/PD} mem/xmm	
		MIN{SS/PS/SD/PD} mem/xmm	

xmm: one operand is a 128-bit SSE2 register
mem/xmm: other operand is in memory or an SSE2 register
{SS} Scalar Single precision FP: one 32-bit operand in a 128-bit register
{PS} Packed Single precision FP: four 32-bit operands in a 128-bit register
{SD} Scalar Double precision FP: one 64-bit operand in a 128-bit register
{PD} Packed Double precision FP, or two 64-bit operands in a 128-bit register
{A} 128-bit operand is aligned in memory
{U} means the 128-bit operand is unaligned in memory
{H} means move the high half of the 128-bit operand
{L} means move the low half of the 128-bit operand

Packed and Scalar Double-Precision Floating-Point Operations



X86 SIMD Intrinsics

intel Intrinsics Guide

mul_pd

Technologies

- [ ] MMX
- [ ] SSE
- [ ] SSE2
- [ ] SSE3
- [ ] SSSE3
- [ ] SSE4.1
- [ ] SSE4.2
- [x] AVX
- [ ] AVX2
- [ ] FMA
- [ ] AVX-512
- [ ] KNC
- [ ] SVML
- [ ] Other

Categories

- [ ] Application-Targeted
- [ ] Arithmetic
- [ ] Bit Manipulation
- [ ] Cast
- [ ] Compare

```
__m256d _mm256_mul_pd (__m256d a, __m256d b)
```

Synopsis

```
__m256d _mm256_mul_pd (__m256d a, __m256d b)
#include "immintrin.h"
Instruction: vmulpd ymm, ymm, ymm
CPUID Flags: AVX
```

← Intrinsic

← assembly instruction

Description

Multiply packed double-precision (64-bit) floating-point elements in a and b, and store the results in dst.

Operation

← 4 parallel multiplies

```
FOR j := 0 to 3
	i := j*64
	dst[i+63:i] := a[i+63:i] * b[i+63:i]
ENDFOR
dst[MAX:256] := 0
```

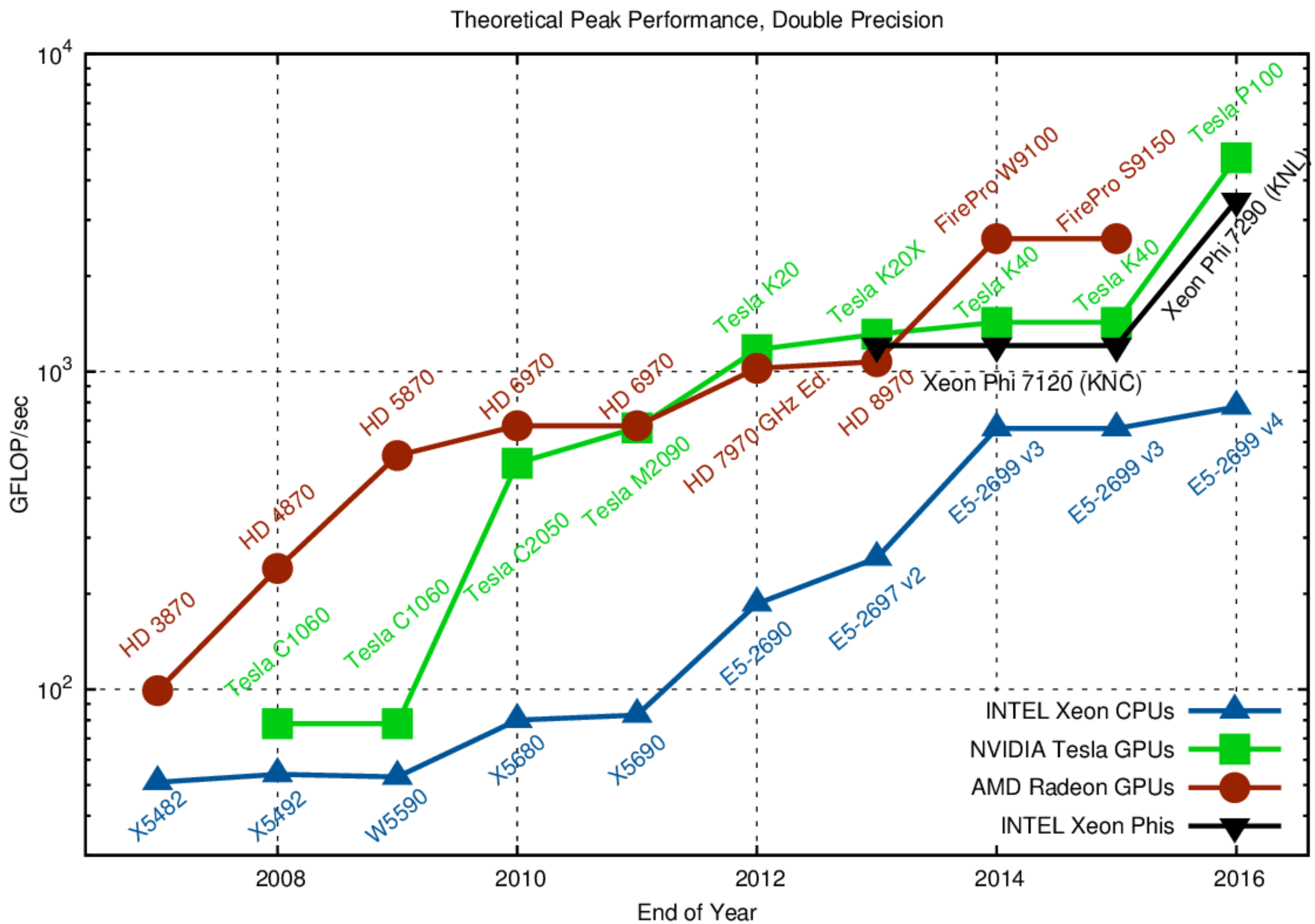
Performance

Architecture	Latency	Throughput
Haswell	5	0.5
Ivy Bridge	5	1
Sandy Bridge	5	1

← 2 instructions per clock cycle (CPI = 0.5)

https://software.intel.com/sites/landingpage/IntrinsicsGuide/

Raw Double-Precision Throughput



https://www.karlrupp.net/2013/06/cpu-gpu-and-mic-hardware-characteristics-over-time/

Example: SIMD Array Processing

```
for each f in array
    f = sqrt(f)

for each f in array
{
    load f to the floating-point register
    calculate the square root
    write the result from the register to memory
}
for each 4 members in array
{
    load 4 members to the SSE register
    calculate 4 square roots in one operation
    store the 4 results from the register to memory
}
```

SIMD style

Data-Level Parallelism and SIMD

- SIMD wants adjacent values in memory that can be operated in parallel
- Usually specified in programs as loops

```
for(i=1000; i>0; i=i-1)
    x[i] = x[i] + s;
```

- How can reveal more data-level parallelism than available in a single iteration of a loop?
- *Unroll loop* and adjust iteration rate

Looping in RISC-V

- D Standard Extension (double) – builds upon F standard extension (float)

Assumptions:

- t1 is initially the address of the element in the array with the highest address
- f0 contains the scalar value s
- 8(t2) is the address of the last element to operate on

CODE:

```
Loop: fld     f2 , 0(t1)      # $f2=array element
      fadd.d f10, f2, f0      # add s to $f2
      fsd     f10, 0(t1)      # store result
      addi    t1, t1, -8      # t1 = t1 -8
      bne     t1, t2, Loop    # repeat loop if t1 != t2
```

Loop Unrolled

```
Loop:
        fld     f2 , 0(t1)
        fadd.d  f10, f2, f0
        fsd     f10, 0(t1)

        fld     f3 , -8(t1)
        fadd.d  f11, f3, f0
        fsd     f11, -8(t1)

        fld     f4 , -16(t1)
        fadd.d  f12, f4, f0
        fsd     f12, -16(t1)

        fld     f5 , -24(t1)
        fadd.d  f13, f5, f0
        fsd     f13, -24(t1)

        addi    t1, t1, -32
        bne     t1, t2, Loop
```

NOTE:

1. Only 1 Loop Overhead every 4 iterations
2. This unrolling works if

   loop_limit(mod 4) = 0
3. Using different registers for each iteration eliminates data hazards in pipeline

Loop Unrolled Scheduled

```
Loop:
        fld     f2 , 0(t1)
        fld     f3 , -8(t1)
        fld     f4 , -16(t1)
        fld     f5 , -24(t1)

        fadd.d  f10, f2, f0
        fadd.d  f11, f3, f0
        fadd.d  f12, f4, f0
        fadd.d  f13, f5, f0

        fsd     f10, 0(t1)
        fsd     f11, -8(t1)
        fsd     f12, -16(t1)
        fsd     f13, -24(t1)

        addi    t1, t1, -32
        bne     t1, t2, Loop
```

4 Loads side-by-side:
Could replace with 4-wide SIMD Load

4 Adds side-by-side:
Could replace with 4-wide SIMD Add

4 Stores side-by-side:
Could replace with 4-wide SIMD Store

Loop Unrolling in C

- Instead of compiler doing loop unrolling, could do it yourself in C

```
for(i=1000; i>0; i=i-1)
    x[i] = x[i] + s;
```

- Could be rewritten

What is downside of doing it in C?

```
for(i=1000; i>0; i=i-4) {
   x[i]   = x[i] + s;
   x[i-1] = x[i-1] + s;
   x[i-2] = x[i-2] + s;
   x[i-3] = x[i-3] + s;
   }
```

Generalizing Loop Unrolling

- A loop of **n iterations**
- **k copies** of the body of the loop
- **Assuming (n mod k) ≠ 0**

  Then we will run the loop with 1 copy of the body **(n mod k)** times and with k copies of the body **floor(n/k)** times

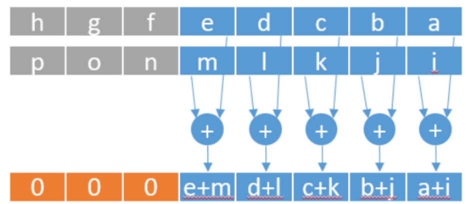
RISC-V Vector Extension

- 32 vector registers
- Need to setup length of data and number of parallel registers to work on before usage (vconfig)!
- vflw.s: vector float load word . stride: load a single word, put in v1 ‘vector length’ times
- vsetvl: ask for certain vector length – hardware knows what it can do (maxvl)!

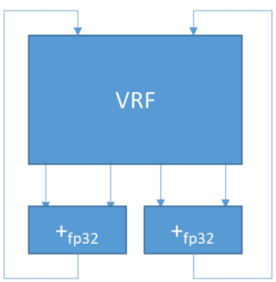
```
    # assume x1 contains size of array
    # assume t1 contains address of array
    # assume x4 contains address of scalar s
    vconfig 0x63          # 4 vregs, 32b data (float)
    vflw.s v1.s, 0(x4)    # load scalar value into v1

loop:
    vsetvl x2, x1         # will set vl and x2 both to min(maxvl, x1)
    vflw v0, 0(t1)        # will load ‘vl’ elements out of ‘vec’
    vfadd.s v2, v1, v0    # do the add
    vsw v2, 0(t1)         # store result back to ‘vec’
    slli x5, x2, 2        # bytes consumed from ‘vec’ (x2 * sizeof(float))
    add t1, t1, x5        # increment ‘vec’ pointer
    sub x1, x1, x2        # subtract from total (x1) work done this iteration (x2)
    bne x1, x0, loop      # if x1 not yet zero, still work to do
```

Hardware Support up to CPU

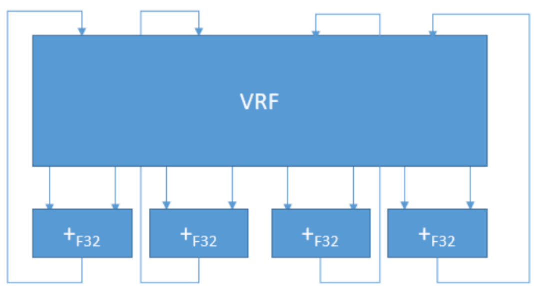


2-lane implementation



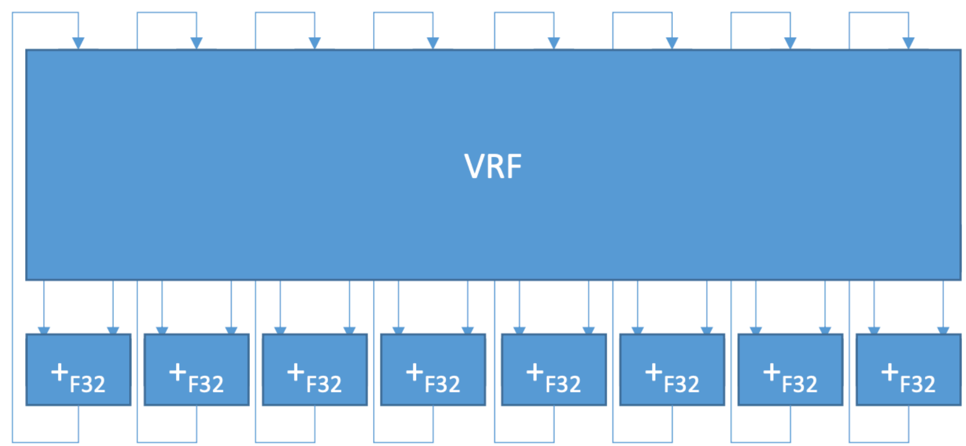
1st clock: a+i, b+j
2nd clock: c+k, d+l
3rd clock: e+m, 0
4th clock: up to you

4-lane implementation



1st clock: a+i, b+j, c+k, d+l
2nd clock: e+m, 0, 0, 0

8-lane implementation (a.k.a. SIMD)



1st clock: a+i, b+j, c+k, d+l, e+m, 0, 0, 0

Number of lanes is transparent to programmer
Same code runs independent of # of lanes

Example: Add Two Single-Precision Floating-Point Vectors

Computation to be performed:

```
vec_res.x = v1.x + v2.x;
vec_res.y = v1.y + v2.y;
vec_res.z = v1.z + v2.z;
vec_res.w = v1.w + v2.w;
```

mov a ps : **mov**e from mem to XMM register, memory **a**ligned, **p**acked **s**ingle precision

add ps : **add** from mem to XMM register, **p**acked **s**ingle precision

mov a ps : **mov**e from XMM register to mem, memory **a**ligned, **p**acked **s**ingle precision

SSE Instruction Sequence:

(Note: Destination on the right in x86 assembly)

```
movaps address-of-v1, %xmm0
       // v1.w | v1.z | v1.y | v1.x -> xmm0
addps address-of-v2, %xmm0
       // v1.w+v2.w | v1.z+v2.z | v1.y+v2.y | v1.x+v2.x -> xmm0
movaps %xmm0, address-of-vec_res
```

Intel SSE Intrinsics

- Intrinsics are C functions and procedures for inserting assembly language into C code, including SSE instructions
  - With intrinsics, can program using these instructions indirectly
  - One-to-one correspondence between SSE instructions and intrinsics

Example SSE Intrinsics

Intrinsics:	Corresponding SSE instructions:
• Vector data type:	
_m128d	
• Load and store operations:	
_mm_load_pd	MOVAPD/aligned, packed double
_mm_store_pd	MOVAPD/aligned, packed double
_mm_loadu_pd	MOVUPD/unaligned, packed double
_mm_storeu_pd	MOVUPD/unaligned, packed double
• Load and broadcast across vector	
_mm_load1_pd	MOVSD + shuffling/duplicating
• Arithmetic:	
_mm_add_pd	ADDPD/add, packed double
_mm_mul_pd	MULPD/multiple, packed double

Example: 2 x 2 Matrix Multiply

Definition of Matrix Multiply:

$$C_{i,j} = (A \times B)_{i,j} = \sum_{k=1}^{2} A_{i,k} \times B_{k,j}$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \times \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1} = A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & C_{1,2} = A_{1,1}B_{1,2} + A_{1,2}B_{2,2} \\ C_{2,1} = A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & C_{2,2} = A_{2,1}B_{1,2} + A_{2,2}B_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} C_{1,1} = 1*1 + 0*2 = 1 & C_{1,2} = 1*3 + 0*4 = 3 \\ C_{2,1} = 0*1 + 1*2 = 2 & C_{2,2} = 0*3 + 1*4 = 4 \end{bmatrix}$$

Example: 2 x 2 Matrix Multiply

Definition of Matrix Multiply:

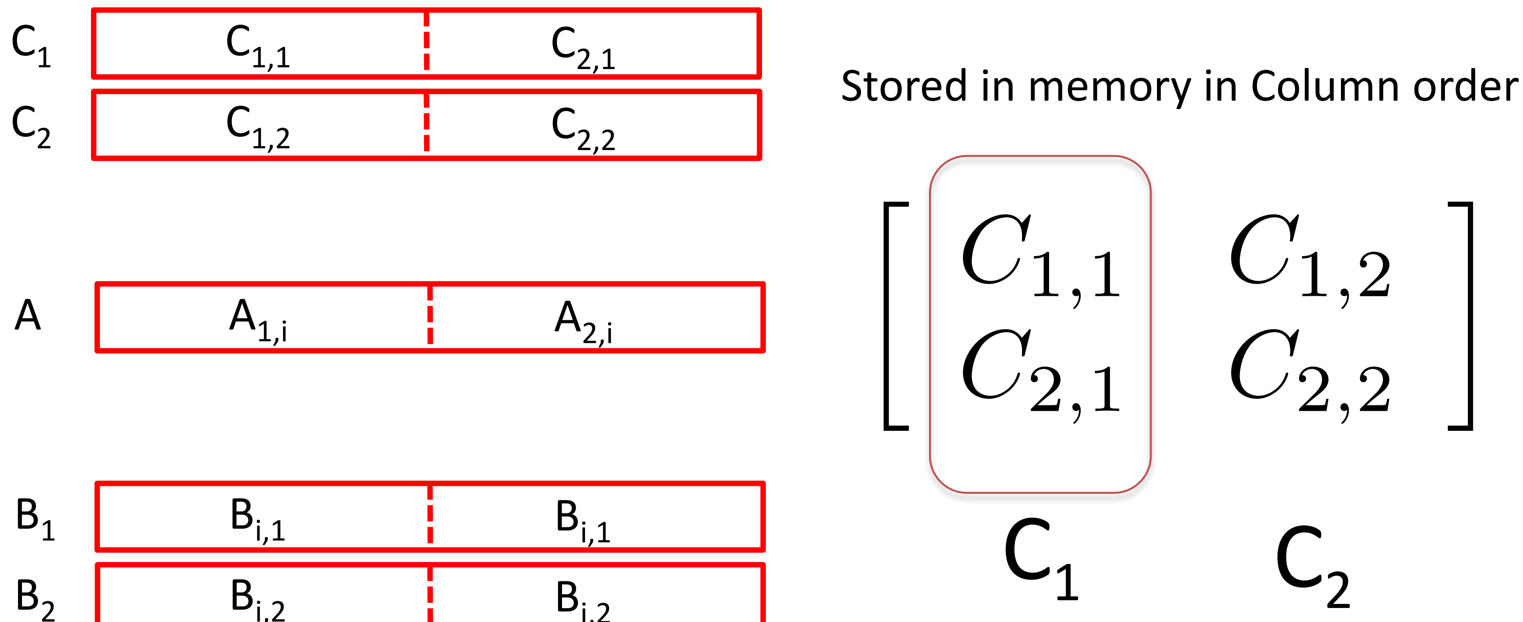
$$C_{i,j} = (A \times B)_{i,j} = \sum_{k=1}^{2} A_{i,k} \times B_{k,j}$$

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \text{x} \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1}=A_{1,1}B_{1,1} + A_{1,2}B_{2,1} & C_{1,2}=A_{1,1}B_{1,2}+A_{1,2}B_{2,2} \\ C_{2,1}=A_{2,1}B_{1,1} + A_{2,2}B_{2,1} & C_{2,2}=A_{2,1}B_{1,2}+A_{2,2}B_{2,2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{x} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} C_{1,1}= 1*1 + 0*2 = 1 & C_{1,2}= 1*3 + 0*4 = 3 \\ C_{2,1}= 0*1 + 1*2 = 2 & C_{2,2}= 0*3 + 1*4 = 4 \end{bmatrix}$$

Example: 2 x 2 Matrix Multiply

- Using the XMM registers
  - 64-bit/double precision/two doubles per XMM reg



Example: 2 x 2 Matrix Multiply

- Initialization

C_1	0	0
C_2	0	0

Example: 2 x 2 Matrix Multiply

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \times \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1}=A_{1,1}B_{1,1}+A_{1,2}B_{2,1} & C_{1,2}=A_{1,1}B_{1,2}+A_{1,2}B_{2,2} \\ C_{2,1}=A_{2,1}B_{1,1}+A_{2,2}B_{2,1} & C_{2,2}=A_{2,1}B_{1,2}+A_{2,2}B_{2,2} \end{bmatrix}$$

- Initialization

C_1	0	0
C_2	0	0

- i = 1

A	$A_{1,1}$	$A_{2,1}$

_mm_load_pd: Load 2 doubles into XMM reg, Stored in memory in Column order

B_1	$B_{1,1}$	$B_{1,1}$
B_2	$B_{1,2}$	$B_{1,2}$

_mm_load1_pd: SSE instruction that loads a double word and stores it in the high and low double words of the XMM register (duplicates value in both halves of XMM)

Example: 2 x 2 Matrix Multiply

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \times \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1}=A_{1,1}B_{1,1}+A_{1,2}B_{2,1} & C_{1,2}=A_{1,1}B_{1,2}+A_{1,2}B_{2,2} \\ C_{2,1}=A_{2,1}B_{1,1}+A_{2,2}B_{2,1} & C_{2,2}=A_{2,1}B_{1,2}+A_{2,2}B_{2,2} \end{bmatrix}$$

- First iteration intermediate result

C_1	$0+A_{1,1}B_{1,1}$	$0+A_{2,1}B_{1,1}$
C_2	$0+A_{1,1}B_{1,2}$	$0+A_{2,1}B_{1,2}$

c1 = _mm_add_pd(c1,_mm_mul_pd(a,b1));
c2 = _mm_add_pd(c2,_mm_mul_pd(a,b2));
SSE instructions first do parallel multiplies and then parallel adds in XMM registers

- i = 1

A	$A_{1,1}$	$A_{2,1}$

_mm_load_pd: Stored in memory in Column order

B_1	$B_{1,1}$	$B_{1,1}$
B_2	$B_{1,2}$	$B_{1,2}$

_mm_load1_pd: SSE instruction that loads a double word and stores it in the high and low double words of the XMM register (duplicates value in both halves of XMM)

Example: 2 x 2 Matrix Multiply

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} \times \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} = \begin{bmatrix} C_{1,1}=A_{1,1}B_{1,1}+A_{1,2}B_{2,1} & C_{1,2}=A_{1,1}B_{1,2}+A_{1,2}B_{2,2} \\ C_{2,1}=A_{2,1}B_{1,1}+A_{2,2}B_{2,1} & C_{2,2}=A_{2,1}B_{1,2}+A_{2,2}B_{2,2} \end{bmatrix}$$

- First iteration intermediate result

C_1	$0+A_{1,1}B_{1,1}$	$0+A_{2,1}B_{1,1}$
C_2	$0+A_{1,1}B_{1,2}$	$0+A_{2,1}B_{1,2}$

c1 = _mm_add_pd(c1,_mm_mul_pd(a,b1));
c2 = _mm_add_pd(c2,_mm_mul_pd(a,b2));
SSE instructions first do parallel multiplies and then parallel adds in XMM registers

- i = 2

A	$A_{1,2}$	$A_{2,2}$

_mm_load_pd: Stored in memory in Column order

B_1	$B_{2,1}$	$B_{2,1}$
B_2	$B_{2,2}$	$B_{2,2}$

_mm_load1_pd: SSE instruction that loads a double word and stores it in the high and low double words of the XMM register (duplicates value in both halves of XMM)

Example: 2 x 2 Matrix Multiply

- Second iteration intermediate result

	$C_{1,1}$	$C_{2,1}$
C_1	$A_{1,1}B_{1,1}+A_{1,2}B_{2,1}$	$A_{2,1}B_{1,1}+A_{2,2}B_{2,1}$
C_2	$A_{1,1}B_{1,2}+A_{1,2}B_{2,2}$	$A_{2,1}B_{1,2}+A_{2,2}B_{2,2}$
	$C_{1,2}$	$C_{2,2}$

c1 = _mm_add_pd(c1,_mm_mul_pd(a,b1));
c2 = _mm_add_pd(c2,_mm_mul_pd(a,b2));
SSE instructions first do parallel multiplies and then parallel adds in XMM registers

- i = 2

A	$A_{1,2}$	$A_{2,2}$

_mm_load_pd: Stored in memory in Column order

B_1	$B_{2,1}$	$B_{2,1}$
B_2	$B_{2,2}$	$B_{2,2}$

_mm_load1_pd: SSE instruction that loads a double word and stores it in the high and low double words of the XMM register (duplicates value in both halves of XMM)

Example: 2 x 2 Matrix Multiply (Part 1 of 2)

```
#include <stdio.h>
// header file for SSE compiler intrinsics
#include <emmintrin.h>

// NOTE: vector registers will be represented in
      // comments as v1 = [ a | b]
// where v1 is a variable of type __m128d and
      // a, b are doubles

int main(void) {
  // allocate A,B,C aligned on 16-byte boundaries
  double A[4] __attribute__ ((aligned (16)));
  double B[4] __attribute__ ((aligned (16)));
  double C[4] __attribute__ ((aligned (16)));
  int lda = 2;
  int i = 0;
  // declare several 128-bit vector variables
  __m128d c1,c2,a,b1,b2;
```

```
// Initialize A, B, C for example
/* A =                    (note column order!)
     1 0
     0 1
   */
  A[0] = 1.0; A[1] = 0.0;  A[2] = 0.0;  A[3] = 1.0;

/* B =                    (note column order!)
     1 3
     2 4
   */
  B[0] = 1.0;  B[1] = 2.0;  B[2] = 3.0;  B[3] = 4.0;

/* C =                    (note column order!)
     0 0
     0 0
   */
  C[0] = 0.0; C[1] = 0.0;  C[2] = 0.0; C[3] = 0.0;
```

Example: 2 x 2 Matrix Multiply (Part 2 of 2)

```
// used aligned loads to set
  // c1 = [c_11 | c_21]
  c1 = _mm_load_pd(C+0*lda);
  // c2 = [c_12 | c_22]
  c2 = _mm_load_pd(C+1*lda);

  for (i = 0; i < 2; i++) {
    /* a =
     i = 0: [a_11 | a_21]
     i = 1: [a_12 | a_22]
    */
    a = _mm_load_pd(A+i*lda);
    /* b1 =
     i = 0: [b_11 | b_11]
     i = 1: [b_21 | b_21]
    */
    b1 = _mm_load1_pd(B+i+0*lda);
    /* b2 =
     i = 0: [b_12 | b_12]
     i = 1: [b_22 | b_22]
    */
    b2 = _mm_load1_pd(B+i+1*lda);
    /* c1 =
     i = 0: [c_11 + a_11*b_11 | c_21 + a_21*b_11]
     i = 1: [c_11 + a_21*b_21 | c_21 + a_22*b_21]
    */
    c1 = _mm_add_pd(c1,_mm_mul_pd(a,b1));
    /* c2 =
     i = 0: [c_12 + a_11*b_12 | c_22 + a_21*b_12]
     i = 1: [c_12 + a_21*b_22 | c_22 + a_22*b_22]
    */
    c2 = _mm_add_pd(c2,_mm_mul_pd(a,b2));
  }

  // store c1,c2 back into C for completion
  _mm_store_pd(C+0*lda,c1);
  _mm_store_pd(C+1*lda,c2);

  // print C
  printf("%g,%g\n%g,%g\n",C[0],C[2],C[1],C[3]);
  return 0;
}
```

DGEMM Speed Comparison

- Double precision GEneral Matrix Multiply: DGEMM
- Intel Core i7-5557U CPU @ 3.10 GHz
  - Instructions per clock (mul_pd) 2; Parallel multiplies per instruction 4
  - => 24.8 GFLOPS
- Python:

```python
def dgemm(N, a, b, c):
    for i in range(N):
        for j in range(N):
            c[i+j*N] = 0
            for k in range(N):
                c[i+j*N] += a[i+k*N] * b[k+j*N]
```

N	Python [Mflops]
32	5.4
160	5.5
480	5.4
960	5.3

- 1 MFLOP = 1 Million floating-point operations per second (fadd, fmul)
- **dgemm**(N …) takes $2*N^3$ flops

C versus Python

- c = a * b
- a, b, c are N x N matrices

```
// Scalar;  P&H p. 226
void dgemm_scalar(int N, double *a, double *b, double *c) {
    for (int i=0;  i<N;  i++)
        for (int j=0;  j<N;  j++) {
            double cij = 0;
            for (int k=0;  k<N;  k++)
                //      a[i][k]  * b[k][j]
                cij += a[i+k*N] * b[k+j*N];
            // c[i][j]
            c[i+j*N] = cij;
        }
}
```

N	C [GFLOPS]	Python [GFLOPS]
32	1.30	0.0054
160	1.30	0.0055
480	1.32	0.0054
960	0.91	0.0053

240x!

Vectorized dgemm

```
// AVX intrinsics;  P&H p. 227
void dgemm_avx(int N, double *a, double *b, double *c) {
    // avx operates on 4 doubles in parallel
    for (int i=0;  i<N;  i+=4) {
        for (int j=0;  j<N;  j++) {
            // c0 = c[i][j]
            __m256d c0 = {0,0,0,0};
            for (int k=0;  k<N;  k++) {
                c0 = _mm256_add_pd(
                        c0,   // c0 += a[i][k] * b[k][j]
                        _mm256_mul_pd(
                            _mm256_load_pd(a+i+k*N),
                            _mm256_broadcast_sd(b+k+j*N)));
            }
            _mm256_store_pd(c+i+j*N, c0); // c[i,j] = c0
        }
    }
}
```

N	Gflops	
	scalar	avx
32	1.30	4.56
160	1.30	5.47
480	1.32	5.27
960	0.91	3.64

- 4x faster
- Still << theoretical 25 GFLOPS

Loop Unrolling

```
// Loop unrolling;  P&H p. 352
const int UNROLL = 4;

void dgemm_unroll(int n, double *A, double *B, double *C) {
    for (int i=0;  i<n;  i+= UNROLL*4) {
        for (int j=0;  j<n;  j++) {
            __m256d c[4];
            for (int x=0;  x<UNROLL;  x++)
                c[x] = _mm256_load_pd(C+i+x*4+j*n);
            for (int k=0;  k<n;  k++) {
                __m256d b = _mm256_broadcast_sd(B+k+j*n);
                for (int x=0;  x<UNROLL;  x++)
                    c[x] = _mm256_add_pd(c[x],
                           _mm256_mul_pd(_mm256_load_pd(A+n*k+x*4+i), b));
            }
            for (int x=0;  x<UNROLL;  x++)
                _mm256_store_pd(C+i+x*4+j*n, c[x]);
        }
    }
}
```

4 registers (arrow pointing to `__m256d c[4];`)

Compiler does the unrolling (arrow pointing to the inner `for (int x=0;  x<UNROLL;  x++)`)

N	GFlops		
	scalar	avx	unroll
32	1.30	4.56	12.95
160	1.30	5.47	19.70
480	1.32	5.27	14.50
960	0.91	3.64	6.91 ?

FPU versus Memory Access

- How many floating-point operations does matrix multiply take?
  - $F = 2 \times N^3$ (N^3 multiplies, N^3 adds)
- How many memory load/stores?
  - $M = 3 \times N^2$ (for A, B, C)
- Many more floating-point operations than memory accesses
  - $q = F/M = 2/3 * N$
  - Good, since arithmetic is faster than memory access
  - Let's check the code …

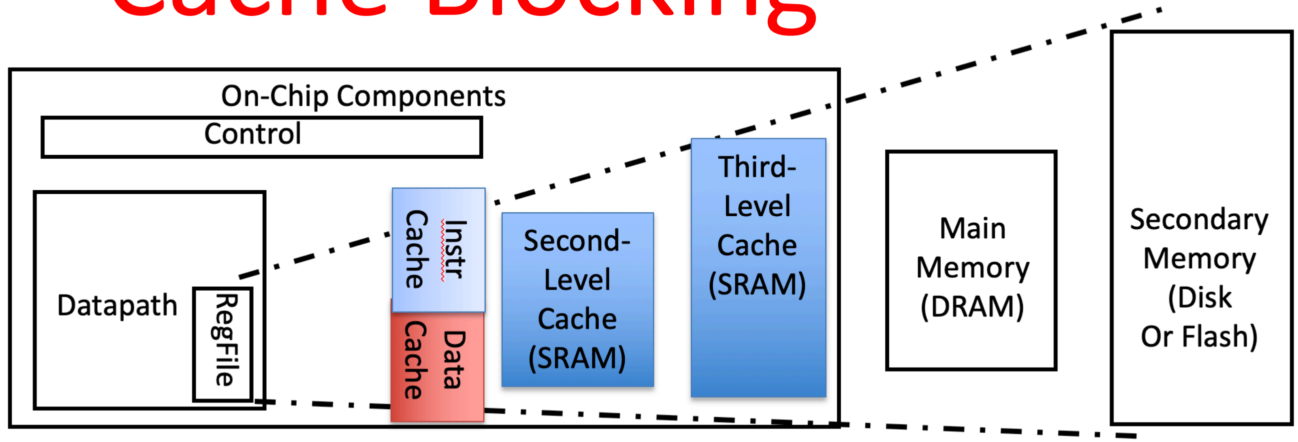
But memory is accessed repeatedly

- q = F/M = 1.6! (1.25 loads and 2 floating-point operations)

Inner loop:

```
for (int k=0;  k<N;  k++) {
    c0 = _mm256_add_pd(
            c0,    // c0 += a[i][k] * b[k][j]
            _mm256_mul_pd(
                _mm256_load_pd(a+i+k*N),
                _mm256_broadcast_sd(b+k+j*N)));
}
```

Cache Blocking

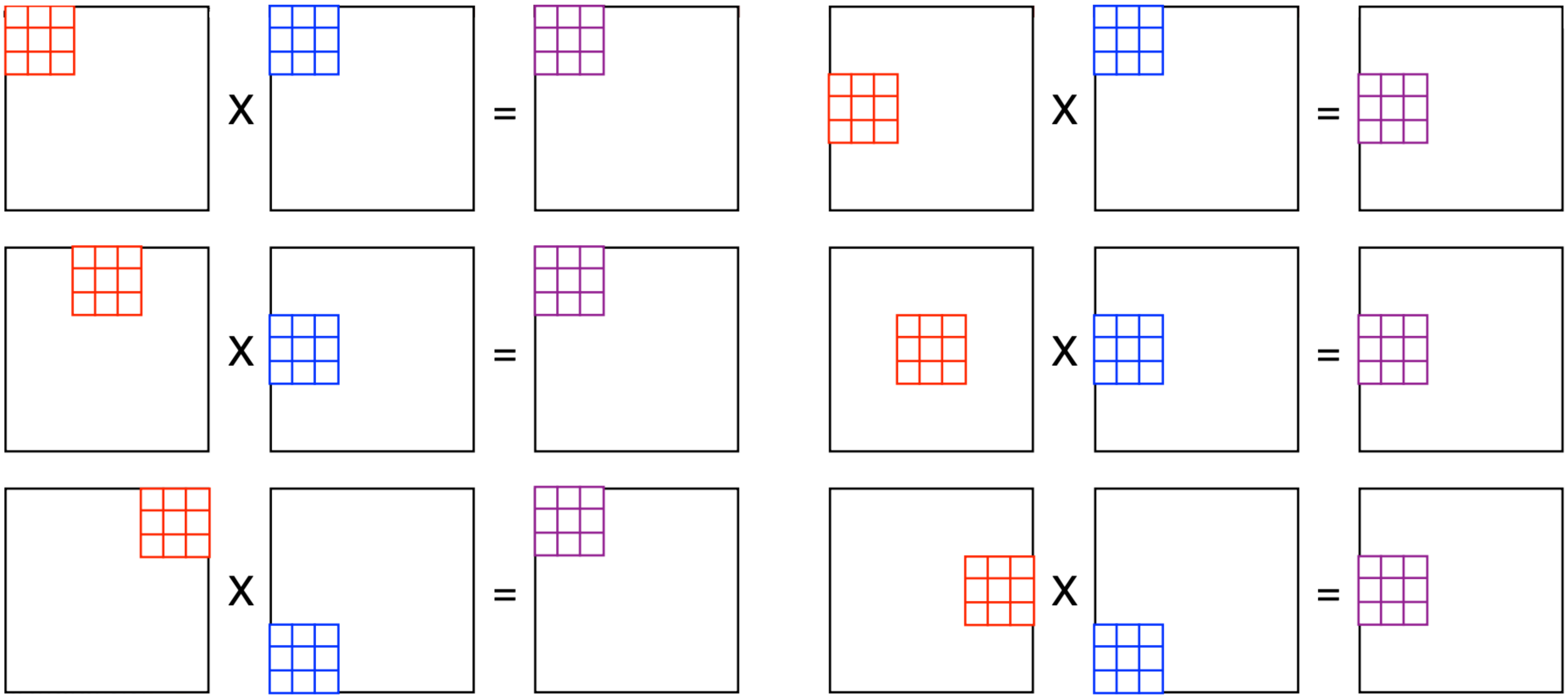


Speed (cycles):	½'s	1's	10's	100's	1,000,000's
Size (bytes):	100's	10K's	M's	G's	T's
Cost/bit:	highest	⟷			lowest

- Where are the operands (A, B, C) stored?
- What happens as N increases?
- Idea: arrange that most accesses are to fast cache!
- Rearrange code to use values loaded in cache many times
- Only "few" accesses to slow main memory (DRAM) per floating point operation
  - -> throughput limited by FP hardware and cache, not slow DRAM

P&H, RISC-V edition p. 465

Blocking Matrix Multiply
(divide and conquer: sub-matrix multiplication)



Memory Access Blocking

```
// Cache blocking;  P&H p. 556
const int BLOCKSIZE = 32;

void do_block(int n, int si, int sj, int sk, double *A, double *B, double *C) {
    for (int i=si;  i<si+BLOCKSIZE;  i+=UNROLL*4)
        for (int j=sj;  j<sj+BLOCKSIZE;  j++) {
            __m256d c[4];
            for (int x=0;  x<UNROLL;  x++)
                c[x] = _mm256_load_pd(C+i+x*4+j*n);
            for (int k=sk;  k<sk+BLOCKSIZE;  k++) {
                __m256d b = _mm256_broadcast_sd(B+k+j*n);
                for (int x=0;  x<UNROLL;  x++)
                    c[x] = _mm256_add_pd(c[x],
                            _mm256_mul_pd(_mm256_load_pd(A+n*k+x*4+i), b));
            }
            for (int x=0;  x<UNROLL;  x++)
                _mm256_store_pd(C+i+x*4+j*n, c[x]);
        }
}

void dgemm_block(int n, double* A, double* B, double* C) {
    for(int sj=0;  sj<n;  sj+=BLOCKSIZE)
        for(int si=0;  si<n;  si+=BLOCKSIZE)
            for (int sk=0;  sk<n;  sk += BLOCKSIZE)
                do_block(n, si, sj, sk, A, B, C);
}
```

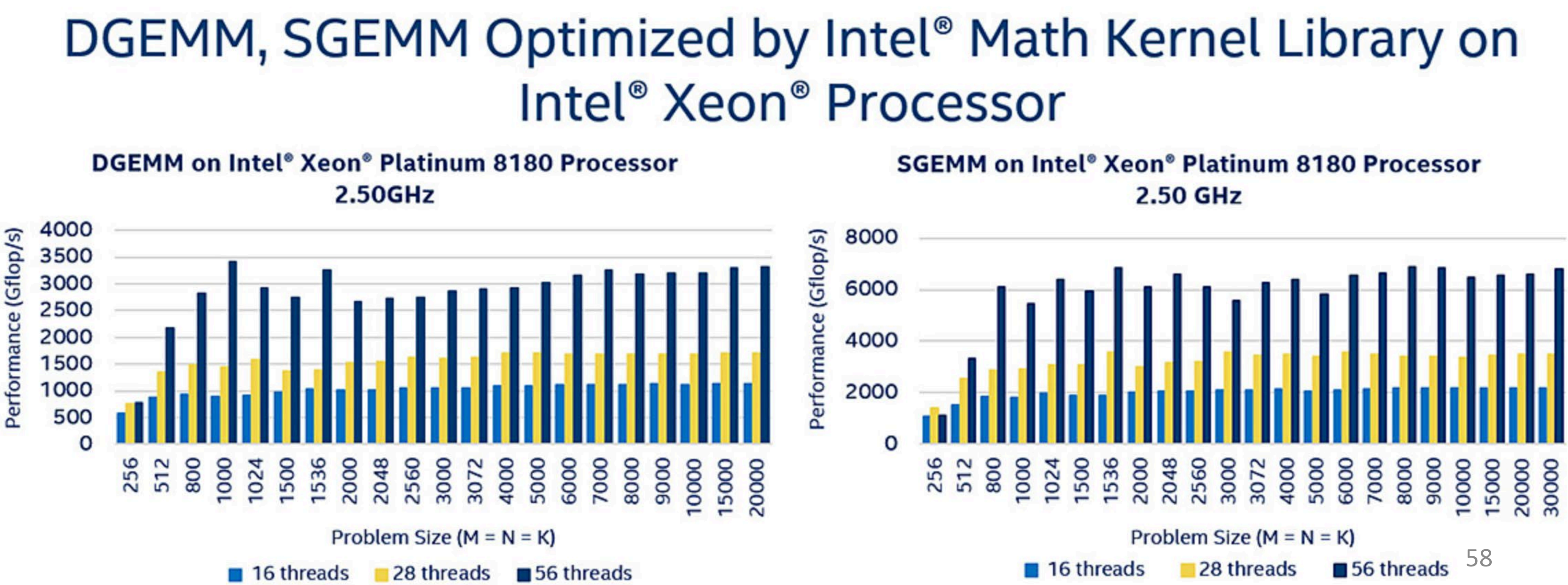
Performance

- Intel i7-5557U theoretical limit (AVX2): 24.8 GFLOPS
- Cache:
  - L3: 4 MB 16-way set associative shared cache
  - L2: 2 x 256 KB 8-way set associative caches
  - L1 Cache: 2 x 32KB 8-way set associative caches (2x: D & I)
- Maximum memory bandwidth (GB/s): 29.9

N	Size	GFlops			
		scalar	**avx**	**unroll**	**blocking**
32	3x 8KiB	1.30	4.56	12.95	13.80
160	3x 200KiB	1.30	5.47	19.70	21.79
480	3x 1.8MiB	1.32	5.27	14.50	20.17
960	3x 7.2MiB	0.91	3.64	6.91	15.82

Intel Math Kernel Library

- AVX programming too hard? Use MKL!
  - C/C++ and Fortran for Windows, Linux, macOS
- Knowledge about AVX still very helpful for using MKL (e.g. Cache blocking, …)
- MKL also for multi-threading…



And in Conclusion, …

- Amdahl's Law: Serial sections limit speedup
- Flynn Taxonomy
- Intel SSE SIMD Instructions
  - Exploit data-level parallelism in loops
  - One instruction fetch that operates on multiple operands simultaneously
  - 128-bit XMM registers
- SSE Instructions in C
  - Embed the SSE machine instructions directly into C programs through use of intrinsics
  - Achieve efficiency beyond that of optimizing compiler