上海科技大学
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CS283: **Robotics Spring 2024: Kinematics**

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KINEMATICS

Motivation

- Autonomous mobile robots move around in the environment. Therefore **ALL** of them:
  - They need to know **where** they **are**.
  - They need to know **where** their **goal** is.
  - They need to know **how** to get there.

- **Odometry!**
- Robot:
  - I know how fast the wheels turned =>
  - I know how the robot moved =>
  - I know where I am ☺

Odometry

- Robot:
  - I know how fast the wheels turned =>
  - I know how the robot moved =>
  - I know where I am ☺
- Marine Navigation: Dead reckoning (using heading sensor)
- Sources of error (AMR pages 269 - 270):
  - Wheel slip
    - Uneven floor contact (non-planar surface)
    - Robot kinematic: tracked vehicles, 4 wheel differential drive..
  - Integration from speed to position: Limited resolution (time and measurement)
  - Wheel misalignment
  - Wheel diameter uncertainty
  - Variation in contact point of wheel

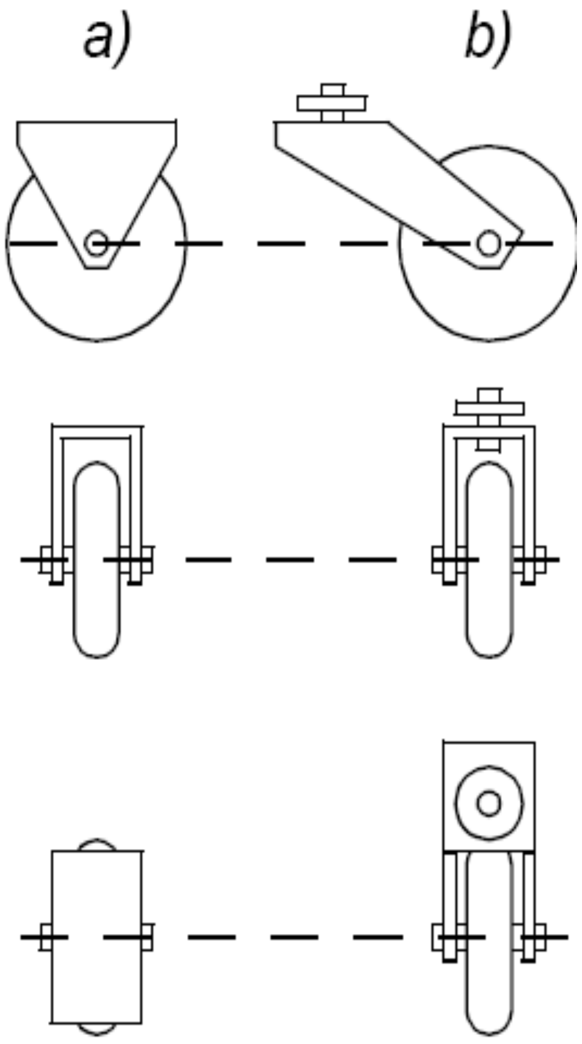
Mobile Robots with Wheels

- Wheels are the most appropriate solution for most applications
- Three wheels are sufficient to guarantee stability
- With more than three wheels an appropriate suspension is required
- Selection of wheels depends on the application

The Four Basic Wheels Types I

- a) Standard wheel: Two degrees of freedom; rotation around the (motorized) wheel axle and the contact point

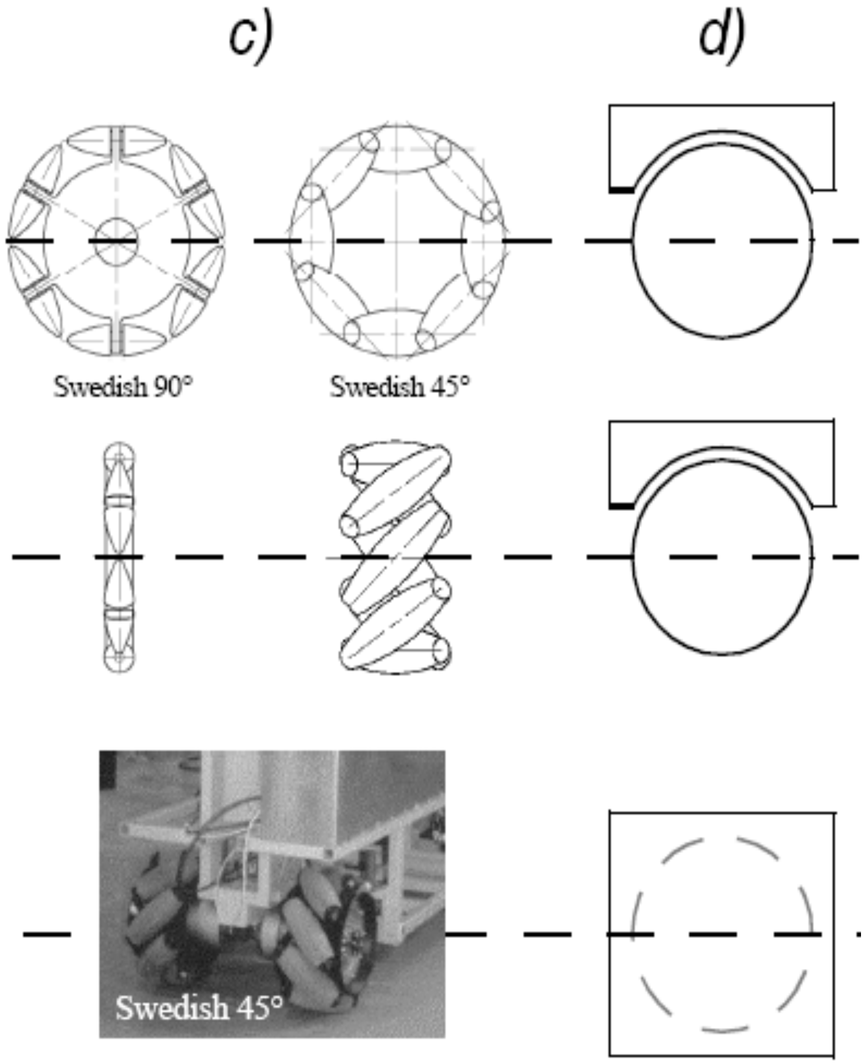
- b) Castor wheel: Three degrees of freedom; rotation around the wheel axle, the contact point and the castor axle



The Four Basic Wheels Types II

- c) Swedish wheel: Three degrees of freedom; rotation around the (motorized) wheel axle, around the rollers and around the contact point

- d) Ball or spherical wheel: Suspension technically not solved

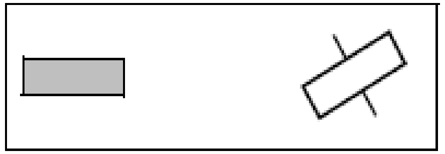


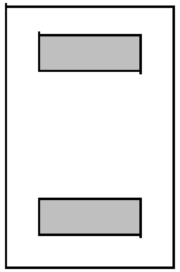
Characteristics of Wheeled Robots and Vehicles

- Stability of a vehicle is be guaranteed with 3 wheels
  - center of gravity is within the triangle with is formed by the ground contact point of the wheels.
- Stability is improved by 4 and more wheel
  - however, this arrangements are hyperstatic and require a flexible suspension system.
- Bigger wheels allow to overcome higher obstacles
  - but they require higher torque or reductions in the gear box.
- Most arrangements are non-holonomic (see chapter 3)
  - require high control effort
- Combining actuation and steering on one wheel makes the design complex and adds additional errors for odometry.

Different Arrangements of Wheels I

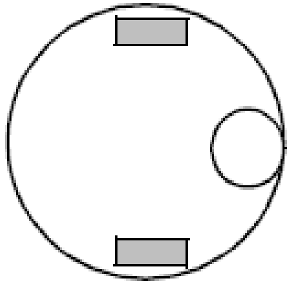
- Two wheels

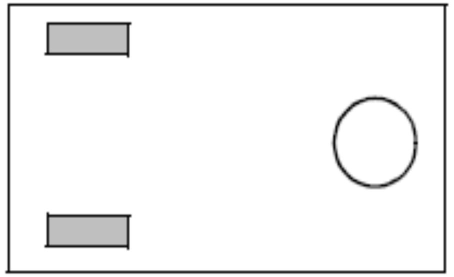


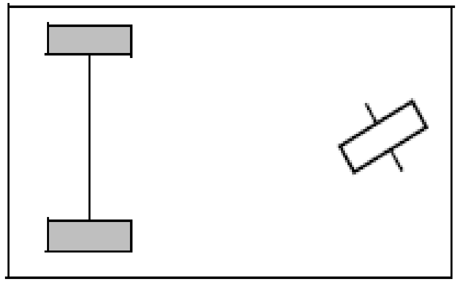


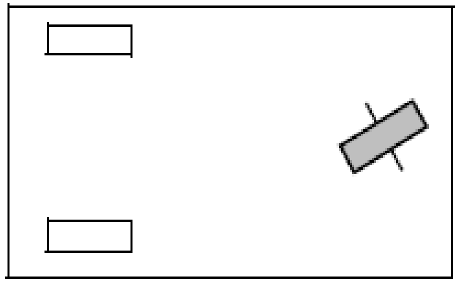
Center of gravity below axle

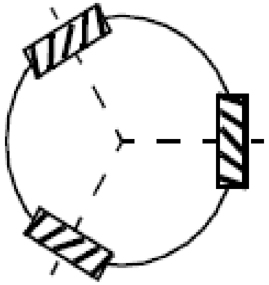
- Three wheels



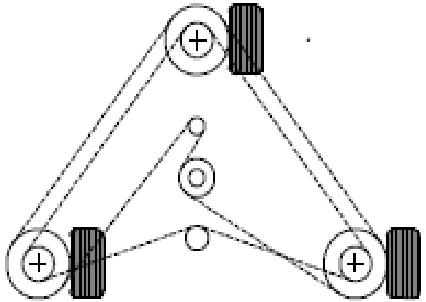








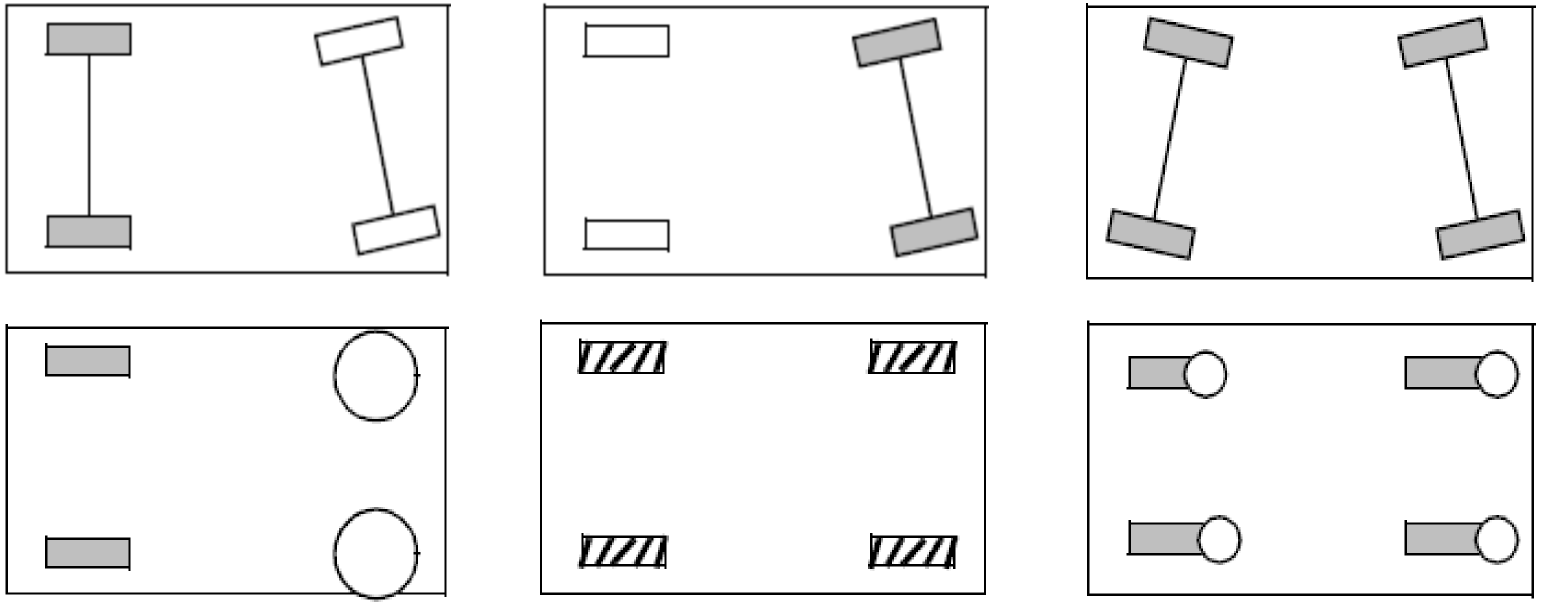
Omnidirectional Drive



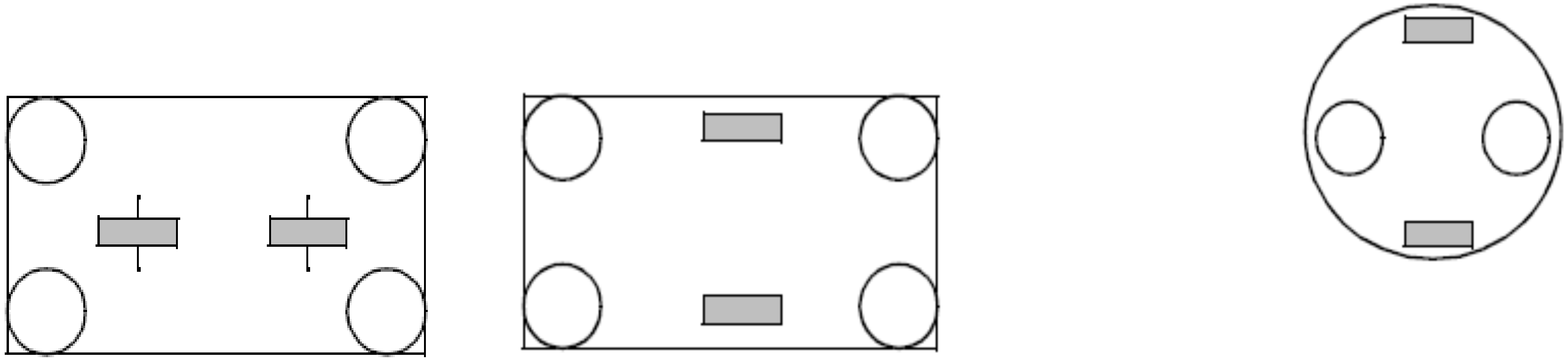
Synchro Drive

Different Arrangements of Wheels II

- Four wheels

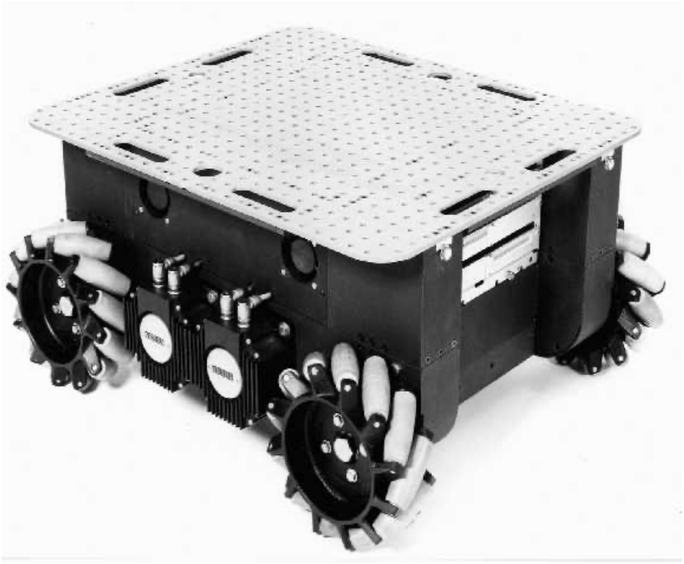


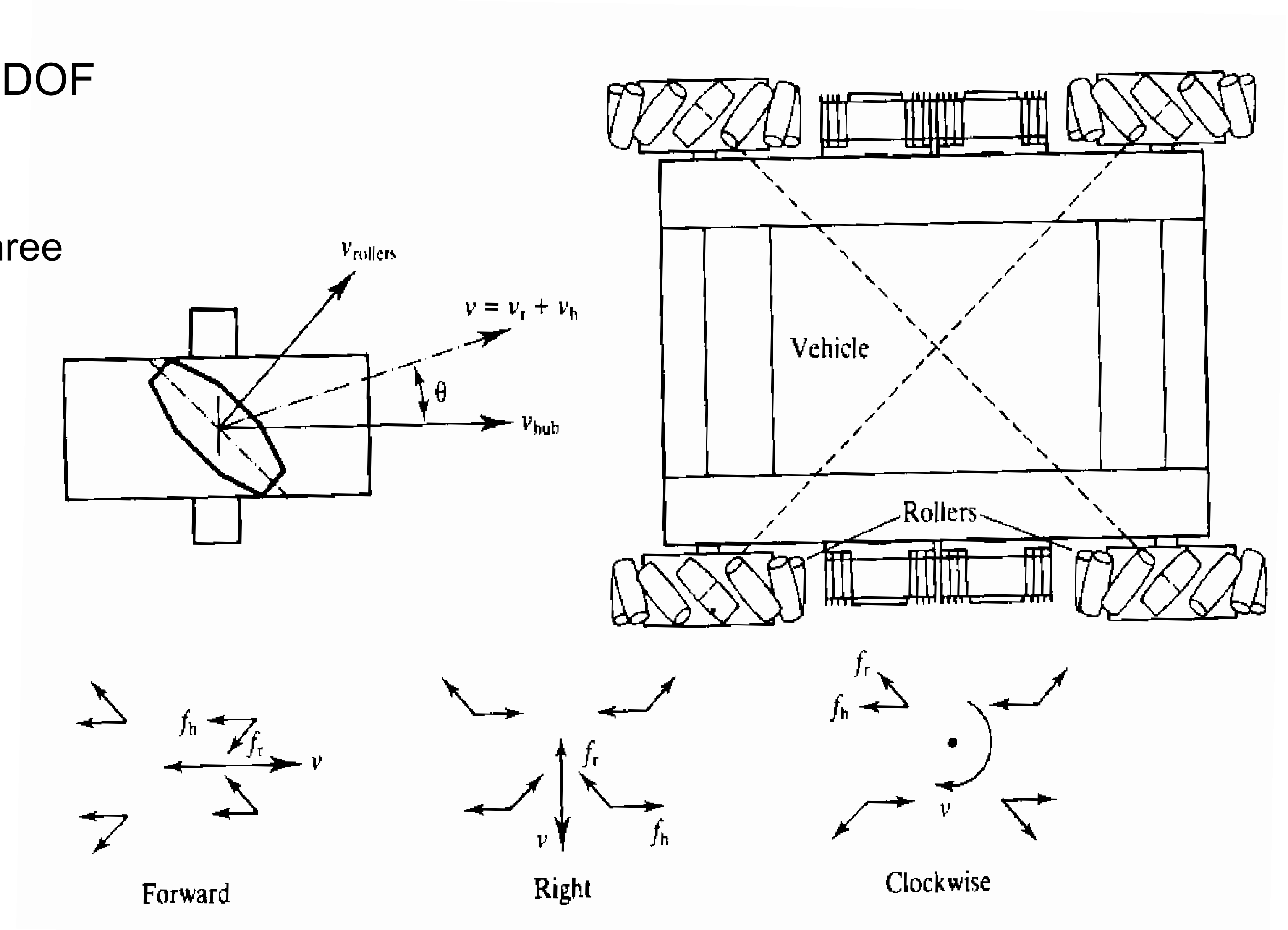
- Six wheels



Uranus, CMU: Omnidirectional Drive with 4 Wheels

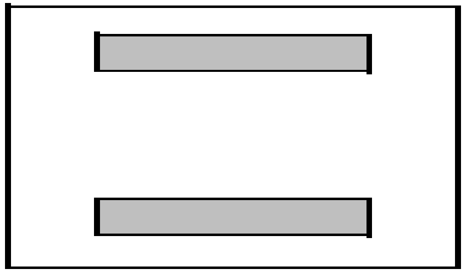
- Movement in the plane has 3 DOF
  - thus only three wheels can be independently controlled
  - It might be better to arrange three swedish wheels in a triangle

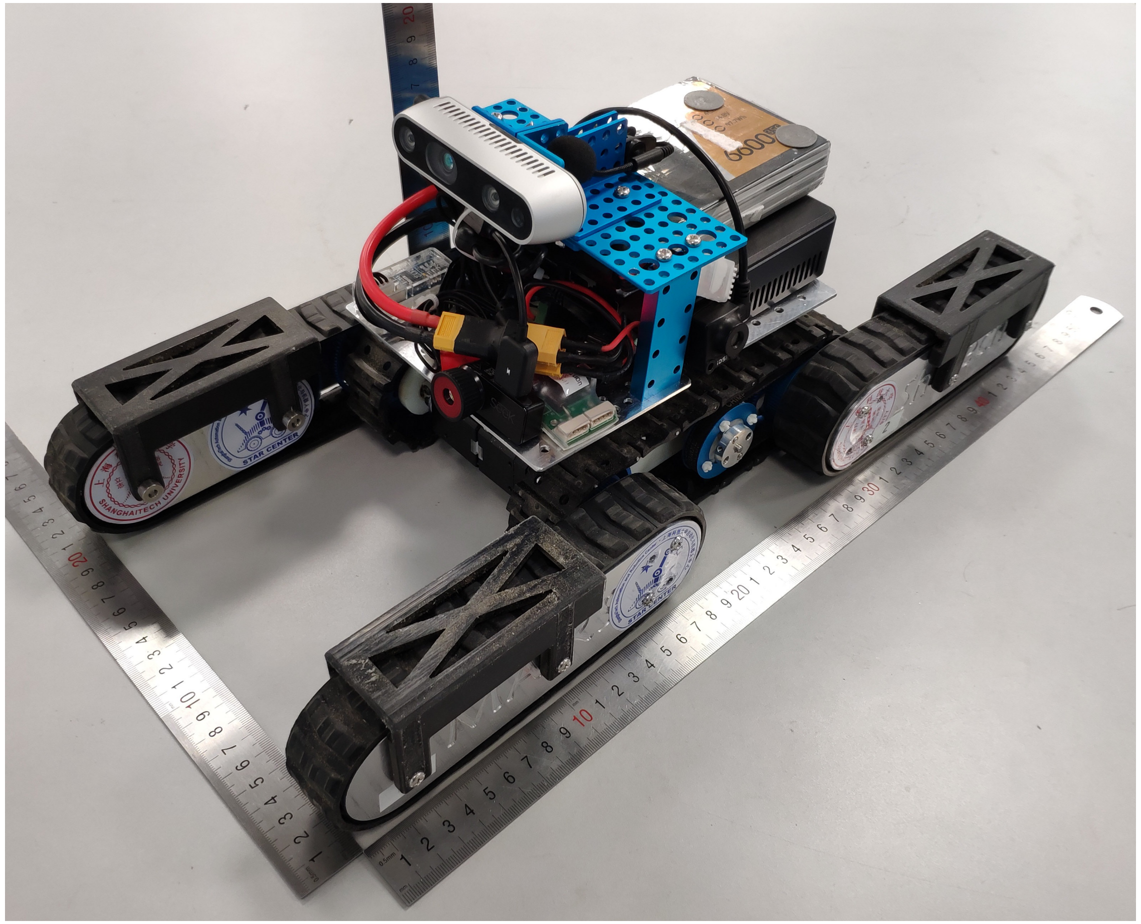




MARS Rescue Robot: Tracked Differential Drive

- Kinematic Simplification:
  - 2 Wheels, located at the center

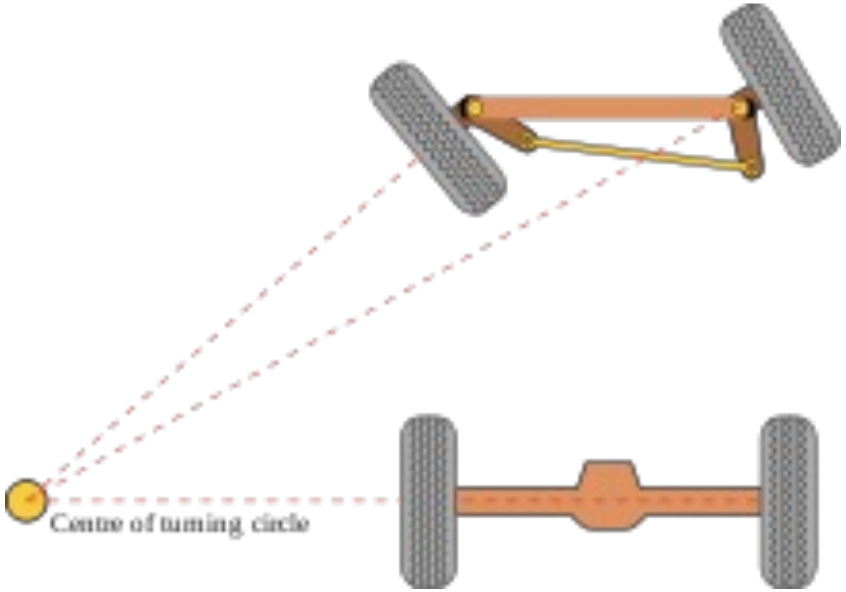




Differential Drive Robots

Ackermann Robot

- No sideways slip than differential drive during turning ☺
- Cannot turn on the spot ☹



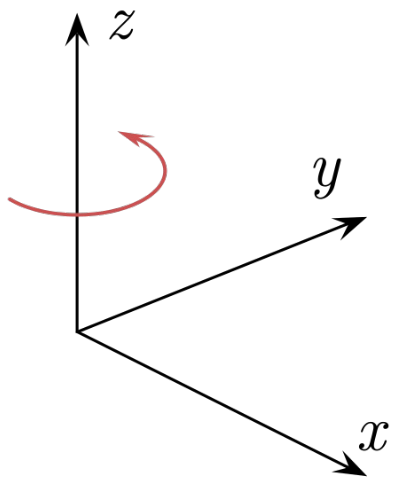
Introduction: Mobile Robot Kinematics

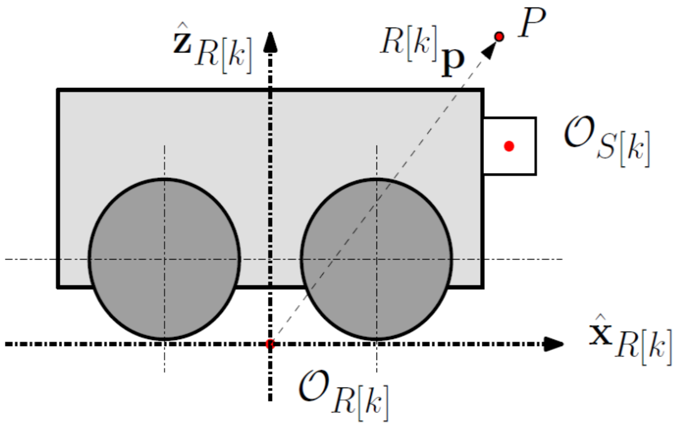
- Aim
  - Description of mechanical behavior of the robot for *design* and *control*
  - Similar to robot manipulator kinematics
  - However, mobile robots can move unbound with respect to its environment
    - there is no direct way to measure the robot’s position
    - Position must be integrated over time
    - Leads to inaccuracies of the position (motion) estimate
      -> the number 1 challenge in mobile robotics

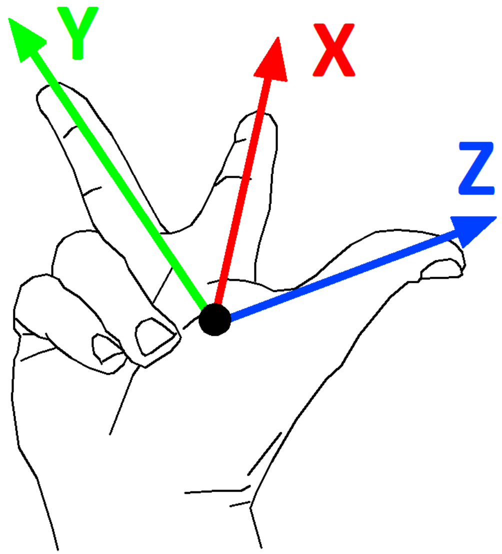
COORDINATE SYSTEM

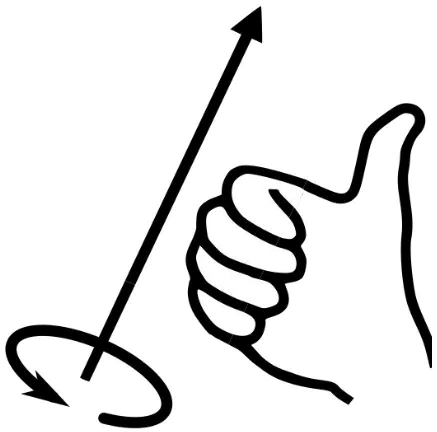
Right Hand Coordinate System

- Standard in Robotics
- Positive rotation around X is anti-clockwise
- Right-hand rule mnemonic:
  - Thumb: z-axis
  - Index finger: x-axis
  - Second finger: y-axis
  - Rotation: Thumb = rotation axis, positive rotation in finger direction
- Robot Coordinate System:
  - X front
  - Z up (Underwater: Z down)
  - Y ???



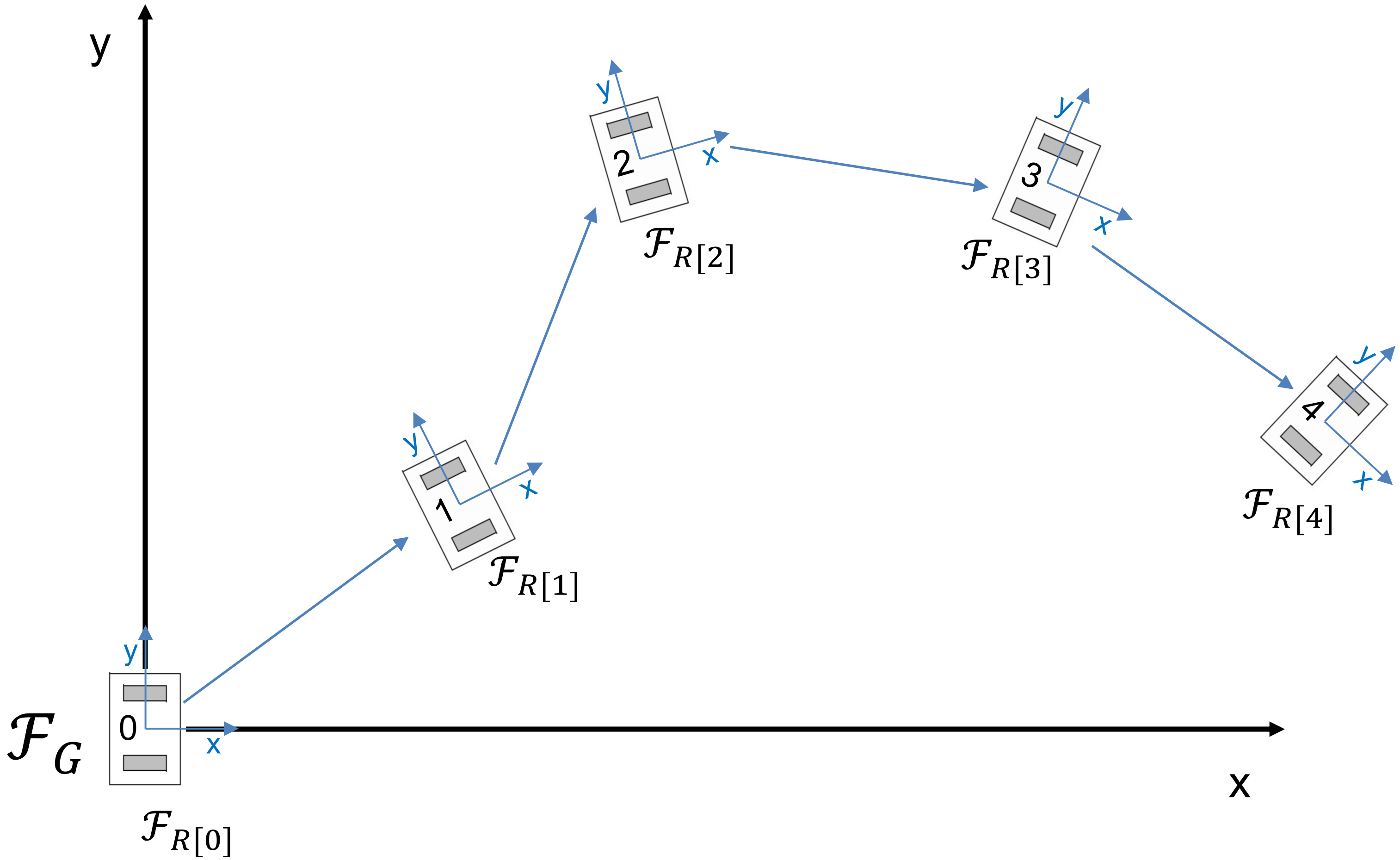






Right Hand Rule http://en.wikipedia.org/wiki/Right-hand_rule

Odometry



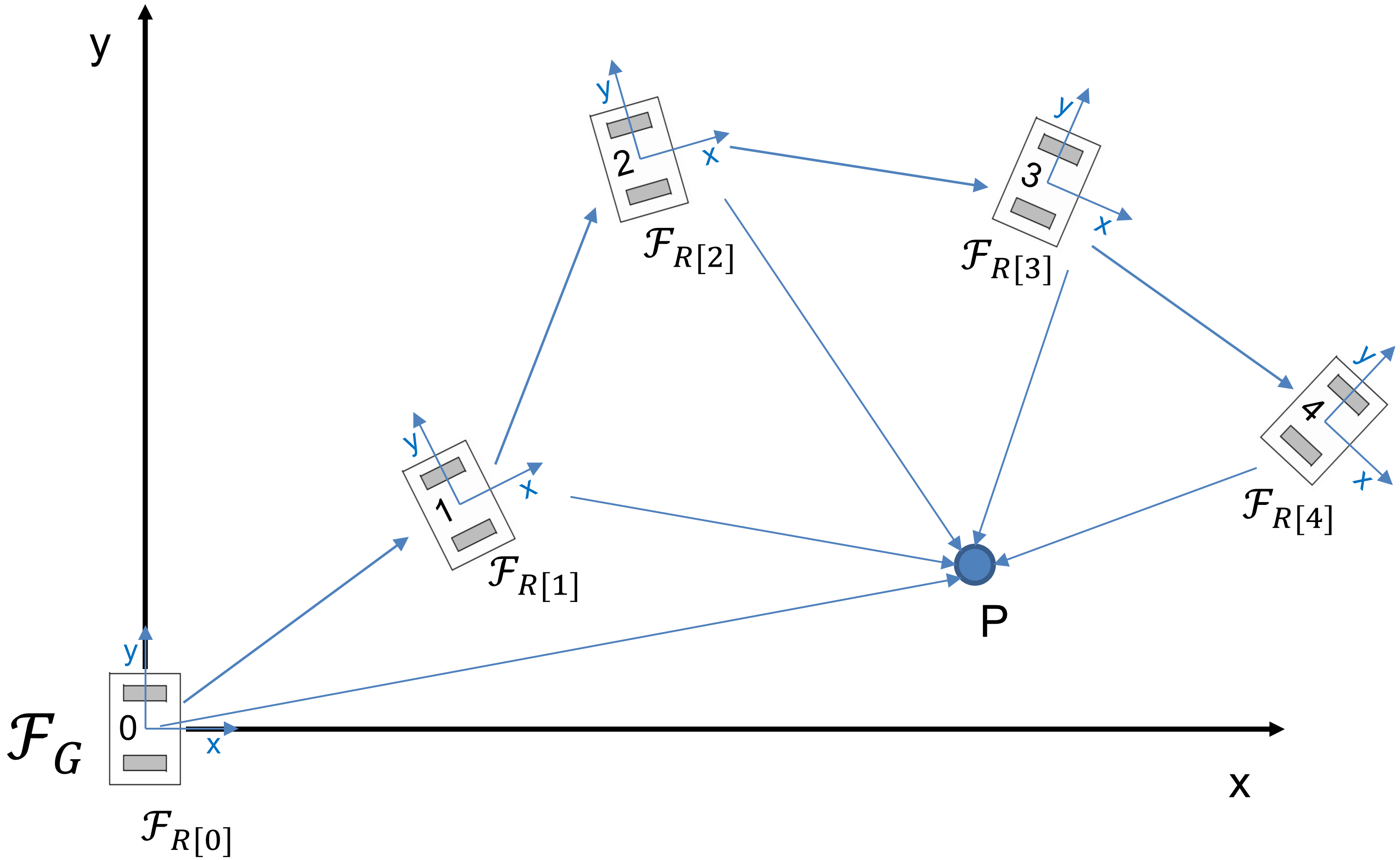
With respect to the robot start pose:
Where is the robot now?

Two approaches – same result:

- Geometry (easy in 2D)
- Transforms (better for 3D)

$\mathcal{F}_{R[X]}$: The ***F***rame of reference (the local coordinate system) of the ***R***obot at the time ***X***

Use of robot frames $\mathcal{F}_{R[X]}$

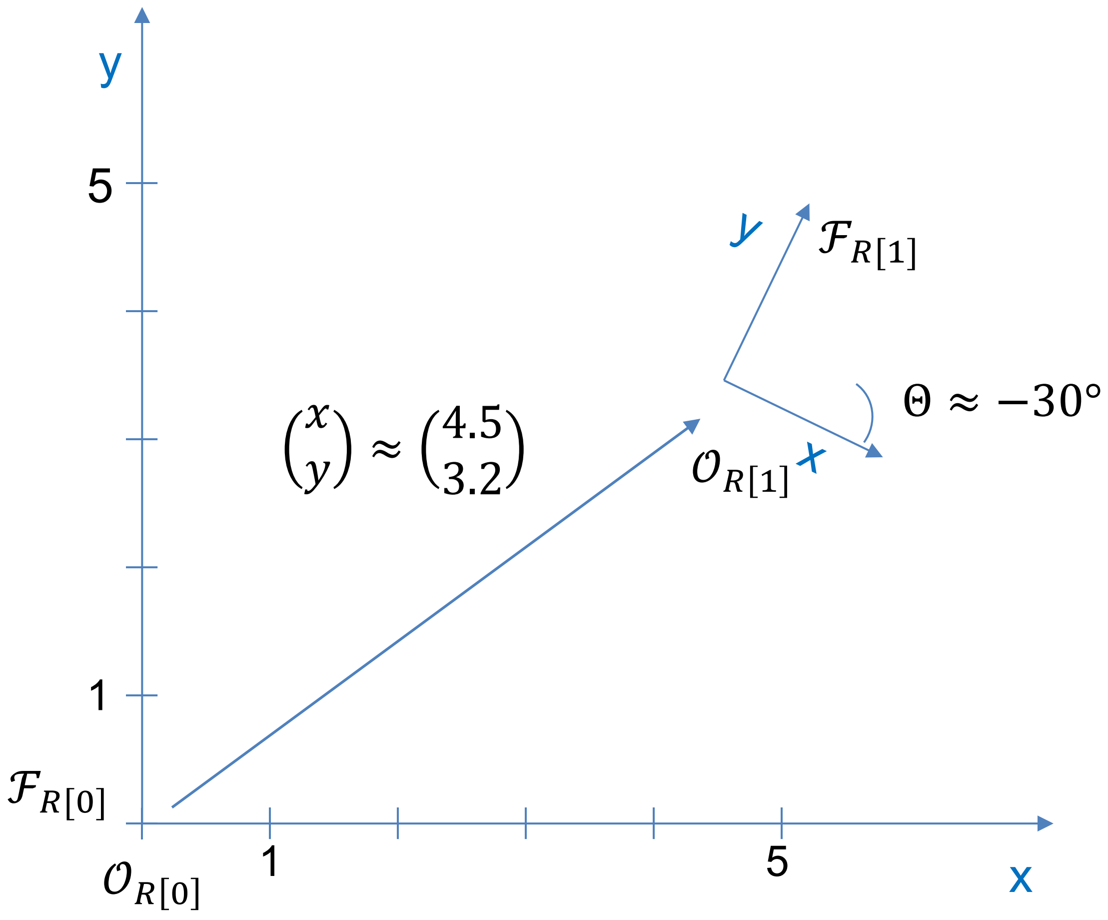


$\mathcal{O}_{R[X]}$: Origin of $\mathcal{F}_{R[X]}$ (coordinates (0, 0))

$\overrightarrow{\mathcal{O}_{R[X]}\mathrm{P}}$: position vector from $\mathcal{O}_{R[X]}$ to point P - $\binom{x}{y}$

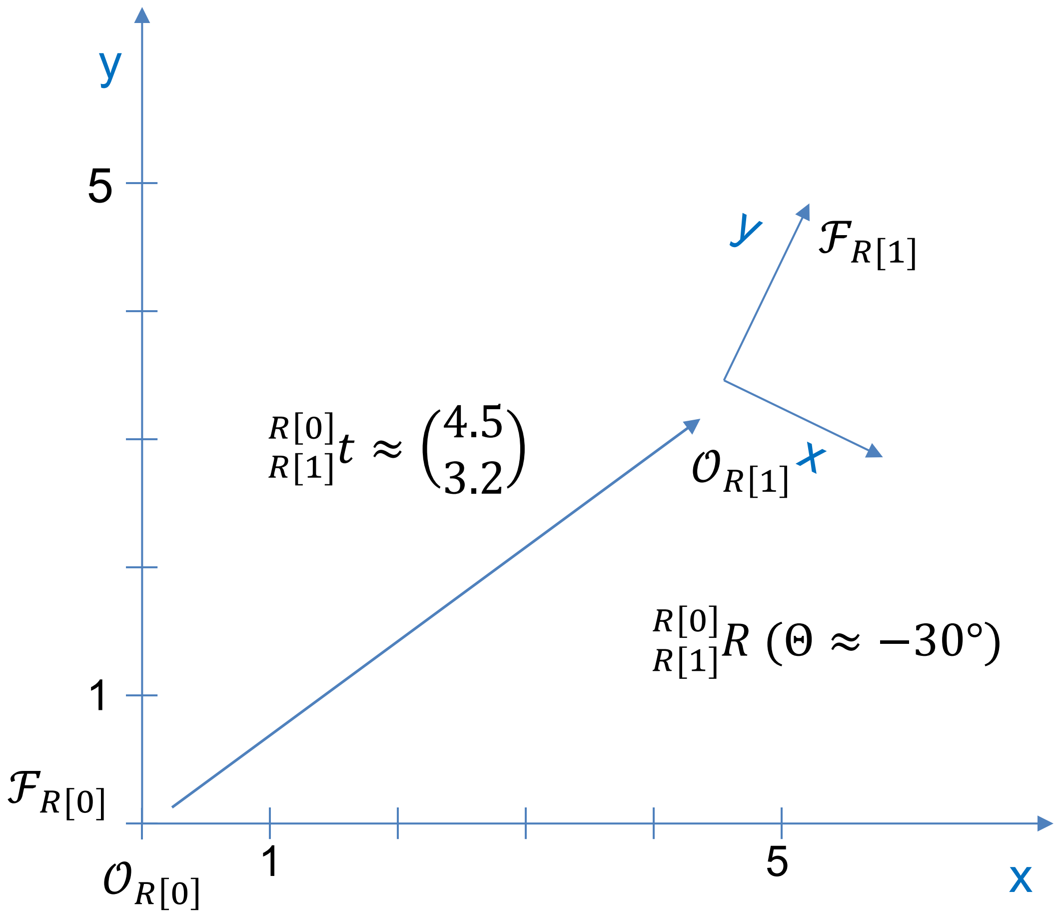
- Object P is observed at times 0 to 4
- Object P is static (does not move)
- The Robot moves (e.g. $\mathcal{F}_{R[0]} \neq \mathcal{F}_{R[1]}$)
- => (x, y) coordinates of P are different in all frames, for example:
  - $\overrightarrow{\mathcal{O}_{R[0]}\mathrm{P}} \neq \overrightarrow{\mathcal{O}_{R[1]}\mathrm{P}}$

Position, Orientation & Pose



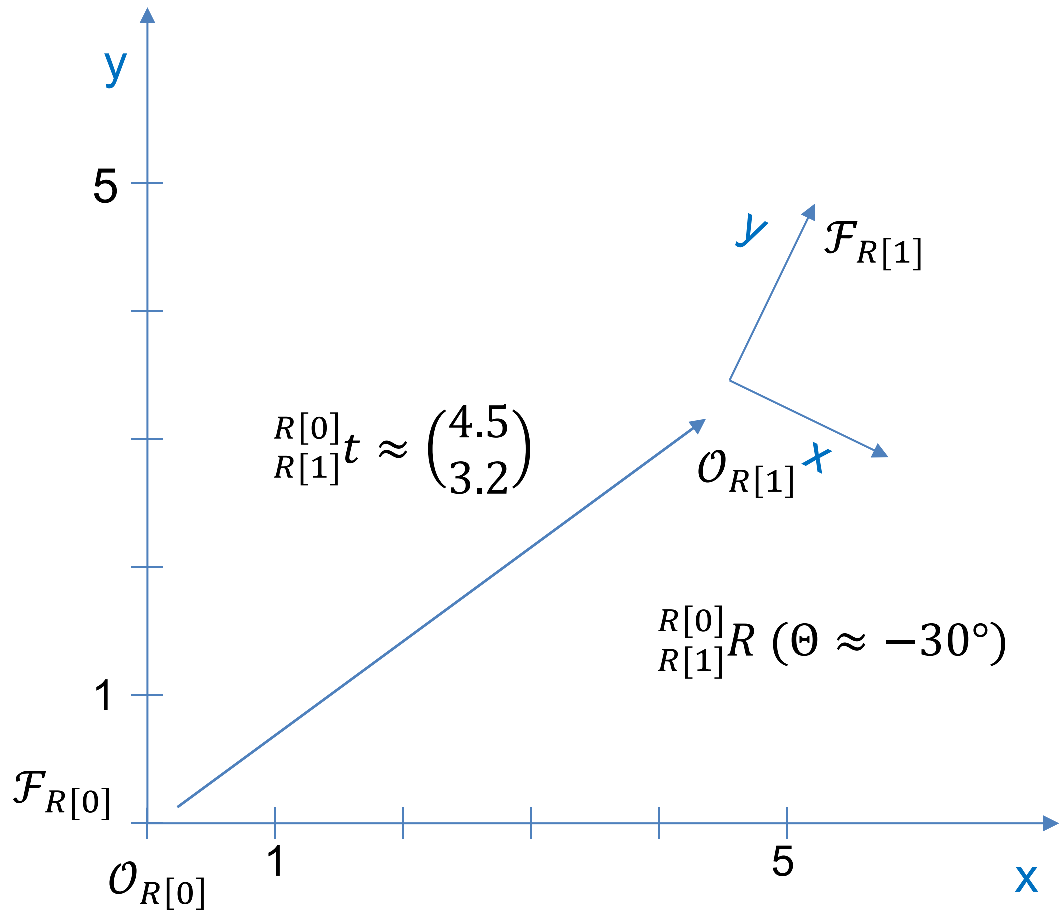
- **Position**:
  - $\binom{x}{y}$ coordinates of any object or point (or another frame)
  - with respect to (wrt.) a specified frame
- **Orientation**:
  - (Θ) angle of any oriented object (or another frame)
  - with respect to (wrt.) a specified frame
- **Pose**:
  - $\begin{pmatrix} x \\ y \\ \Theta \end{pmatrix}$ position and orientation of any oriented object
  - with respect to (wrt.) a specified frame

Translation, Rotation & Transform



- **Translation**:
  - $\begin{pmatrix} x \\ y \end{pmatrix}$ difference, change, motion from one reference frame to another reference frame
- **Rotation**:
  - (Θ) difference in angle, rotation between one reference frame and another reference frame
- **Transform**:
  - $\begin{pmatrix} x \\ y \\ \Theta \end{pmatrix}$ difference, motion between one reference frame and another reference frame

Position & Translation, Orientation & Rotation



- $\mathcal{F}_{R[X]}$: Frame of reference of the robot at time X
- Where is that frame $\mathcal{F}_{R[X]}$?
  - Can only be expressed with respect to (wrt.) another frame (e.g. global Frame $\mathcal{F}_G$) =>
  - Pose of $\mathcal{F}_{R[X]}$ wrt. $\mathcal{F}_G$

- $\mathcal{O}_{R[X]}$: Origin of $\mathcal{F}_{R[X]}$
  - $\overrightarrow{\mathcal{O}_{R[X]}\mathcal{O}_{R[X+1]}}$: **Position** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

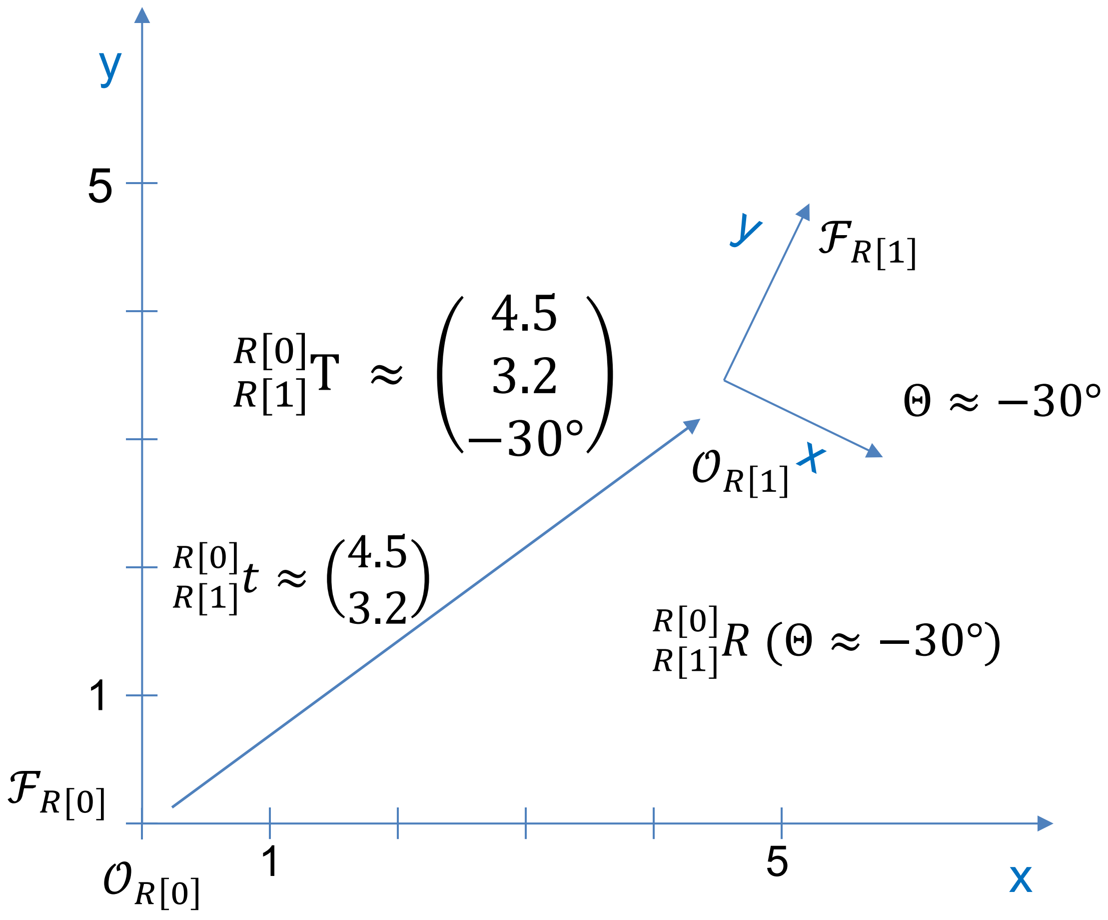
    so $\mathcal{O}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

    $\triangleq \ {}^{R[X]}_{R[X+1]}t$: **Translation**

- The angle Θ between the x-Axes:
  - **Orientation** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

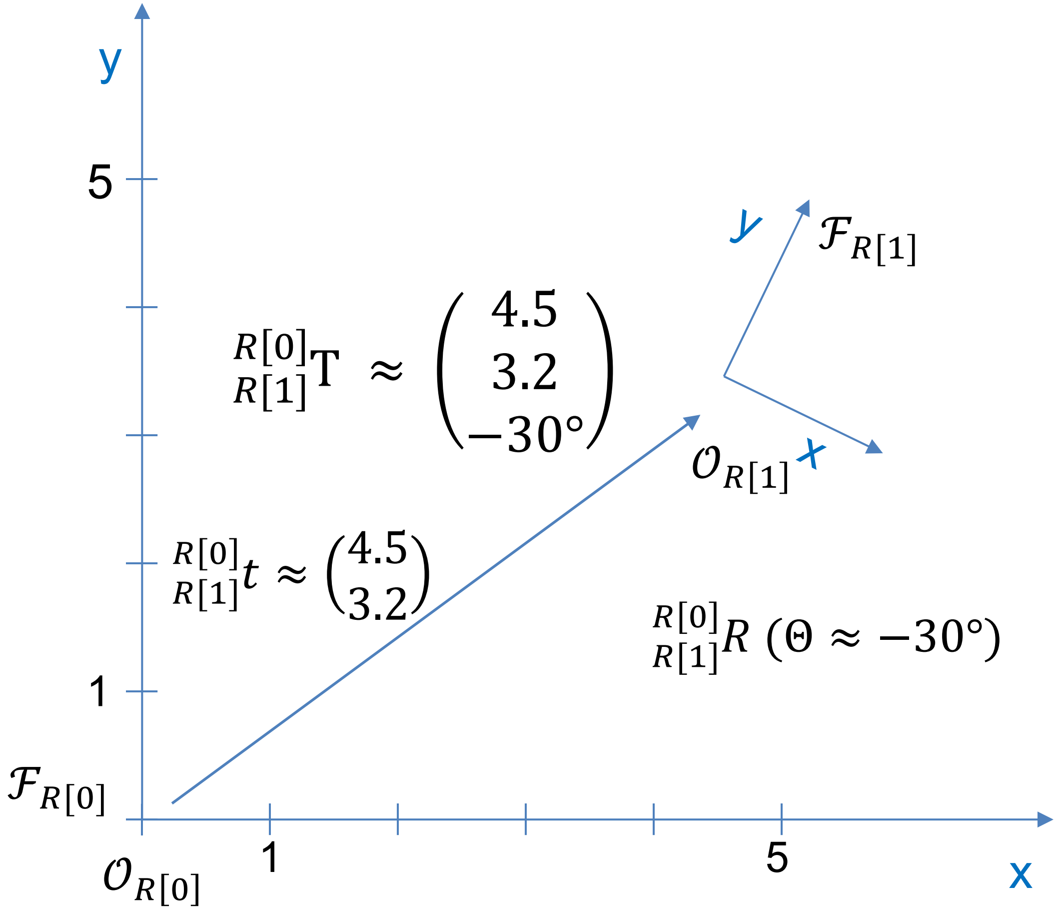
    $\triangleq \ {}^{R[X]}_{R[X+1]}R$: **Rotation** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

Transform



- $^{R[X]}_{R[X+1]}t$: **Translation**
  - Position vector (x, y) of $R[X+1]$ wrt. $R[X]$
- $^{R[X]}_{R[X+1]}R$: **Rotation**
  - Angle (Θ) of $R[X+1]$ wrt. $R[X]$
- **Transform:** $^{R[X]}_{R[X+1]}\mathrm{T} \equiv \begin{Bmatrix} ^{R[X]}_{R[X+1]}t \\ ^{R[X]}_{R[X+1]}R \end{Bmatrix}$

Geometry approach to Odometry



We want to know:

- Position of the robot (x, y)
- Orientation of the robot (Θ)
- => together: Pose $\begin{pmatrix} x \\ y \\ \Theta \end{pmatrix}$

With respect to (wrt.) $\mathcal{F}_G$: The global frame; global coordinate system

$$\mathcal{F}_{R[0]} = \mathcal{F}_G \Rightarrow \ {}^{G}\mathcal{F}_{R[0]} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

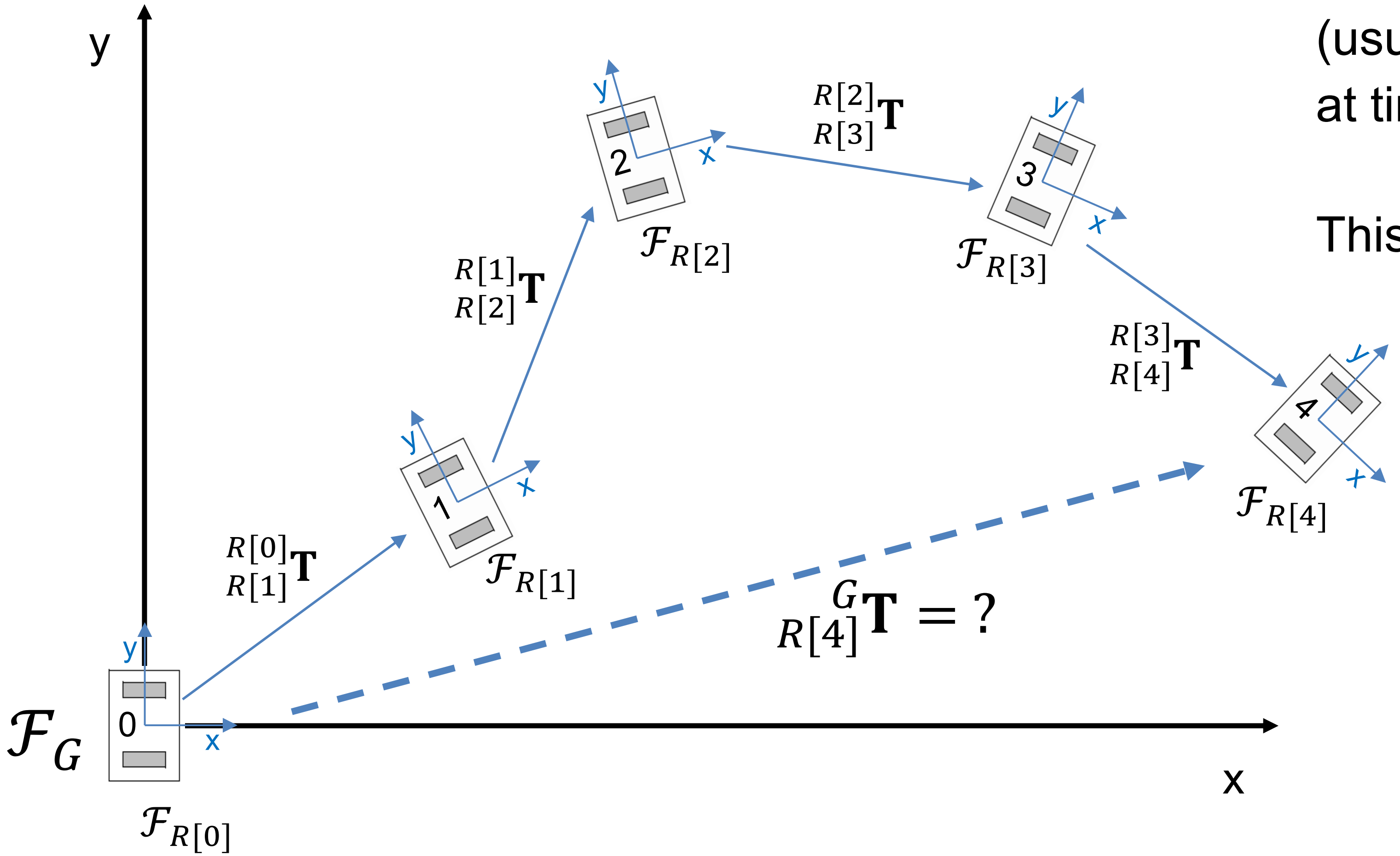
$$^{G}\mathcal{F}_{R[1]} = {}^{R[0]}_{R[1]}\mathrm{T} \approx \begin{pmatrix} 4.5 \\ 3.2 \\ 30° \end{pmatrix}$$

Mathematical approach: Transforms

Where is the Robot now?

The pose of $\mathcal{F}_{R[X]}$ with respect to $\mathcal{F}_G$ (usually = $\mathcal{F}_{R[0]}$) is the pose of the robot at time X.

This is equivalent to ${}^{G}_{R[X]}\mathbf{T}$



Chaining of Transforms

$$ {}^{G}_{R[X+1]}\mathbf{T} = {}^{G}_{R[X]}\mathbf{T} \; {}^{R[X]}_{R[X+1]}\mathbf{T} $$

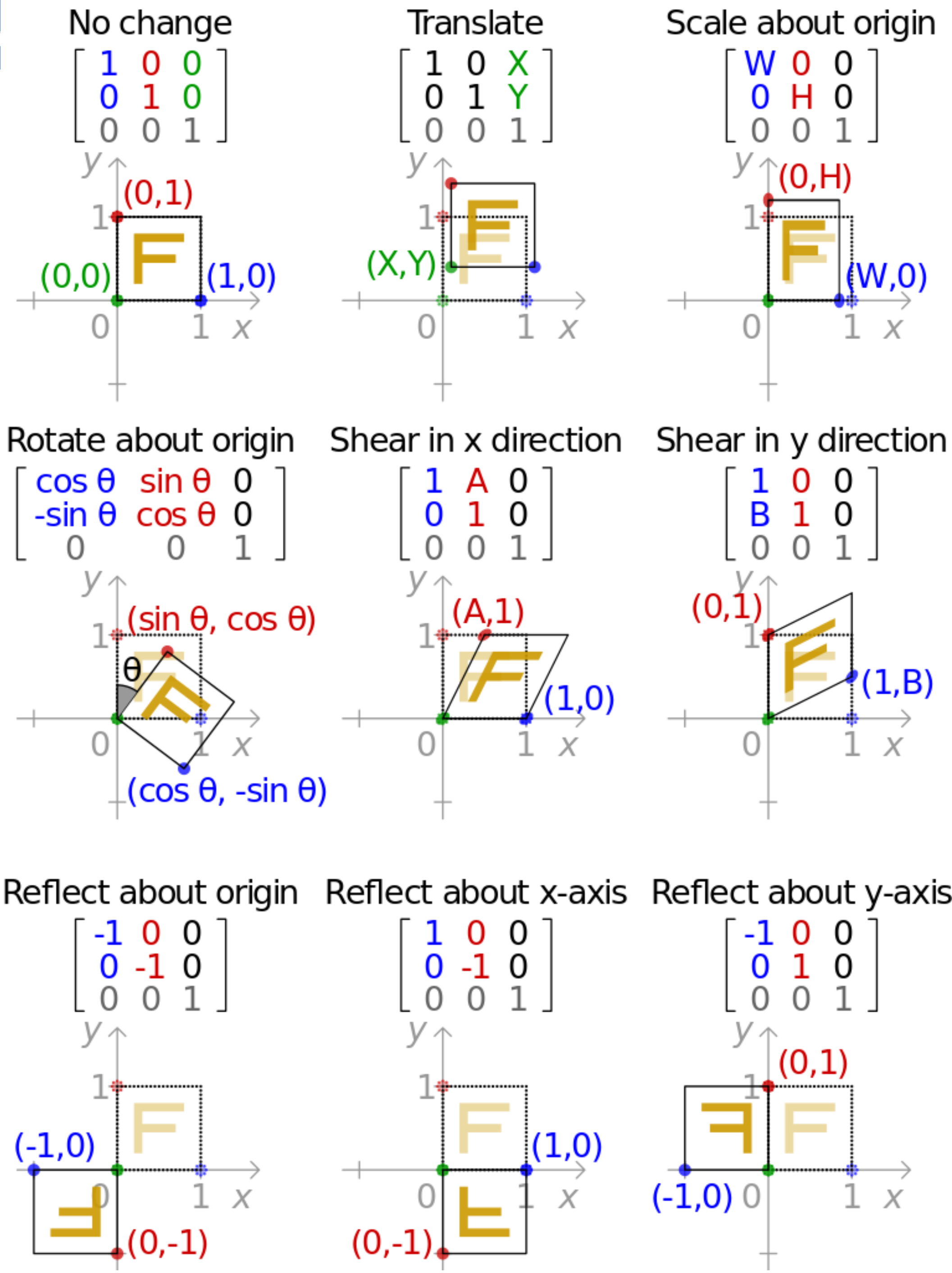
often: $\mathcal{F}_G \equiv \mathcal{F}_{R[0]} \Rightarrow {}^{G}_{R[0]}\mathbf{T} = id$

TRANSFORMS & STUFF ☺

Affine Transformation

- Function between affine spaces. Preserves:
  - points,
  - straight lines
  - planes
  - sets of parallel lines remain parallel
- Allows:
  - Interesting for Robotics: translation, rotation, (scaling), and chaining of those
  - Not so interesting for Robotics: reflection, shearing, homothetic transforms

- Rotation and Translation: $\begin{bmatrix} \cos\theta & \sin\theta & X \\ -\sin\theta & \cos\theta & Y \\ 0 & 0 & 1 \end{bmatrix}$



Math: Rigid Transformation

- Geometric transformation that preserves Euclidean distance between pairs of points.
- Includes reflections (i.e. change from right-hand to left-hand coorinate system and back)
- Just rotation & translation: rigid motions or proper rigid transformations:
  - Decomposed to rotation and translation
  - => subset of Affine Transofrmations

- In Robotics: Just use term **Transform** or **Transformation** for rigid motions (without reflections)

Lie groups for transformations

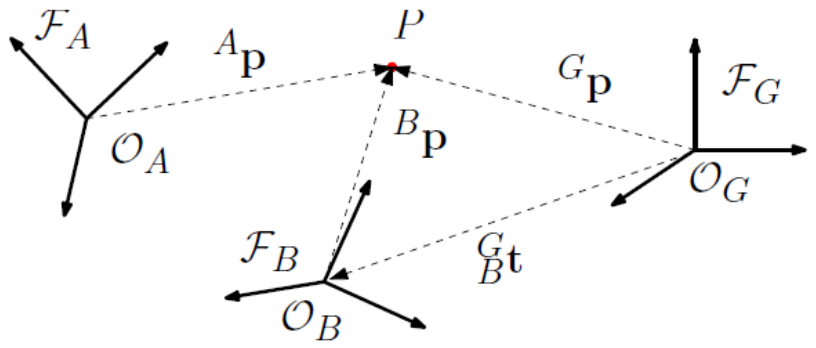
- Smoothly differentiable Group
- No singularities
- Good interpolation

- SO: Special Orthorgonal group
- SE: Special Euclidian group
- Sim_ilarity transform group

Group	Description	Dim.	Matrix Representation
SO(3)	3D Rotations	3	3D rotation matrix
SE(3)	3D Rigid transformations	6	Linear transformation on homogeneous 4-vectors
SO(2)	2D Rotations	1	2D rotation matrix
SE(2)	2D Rigid transformations	3	Linear transformation on homogeneous 3-vectors
Sim(3)	3D Similarity transformations (rigid motion + scale)	7	Linear transformation on homogeneous 4-vectors

http://ethaneade.com/lie.pdf

Transform



Notation	Meaning
$\mathcal{F}_{\mathrm{R}[k]}$	Coordinate frame attached to object 'R' (usually the robot) at sample time-instant k.
$\mathcal{O}_{\mathrm{R}[k]}$	Origin of $\mathcal{F}_{\mathrm{R}[k]}$.
$^{\mathrm{R}[k]}\mathbf{p}$	For any general point P, the position vector $\overrightarrow{\mathcal{O}_{\mathrm{R}[k]}P}$ resolved in $\mathcal{F}_{\mathrm{R}[k]}$.
$^{\mathrm{H}}\hat{\mathbf{x}}_{\mathrm{R}}$	The x-axis direction of $\mathcal{F}_{\mathrm{R}}$ resolved in $\mathcal{F}_{\mathrm{H}}$. Similarly, $^{\mathrm{H}}\hat{\mathbf{y}}_{\mathrm{R}}$, $^{\mathrm{H}}\hat{\mathbf{z}}_{\mathrm{R}}$ can be defined. Obviously, $^{\mathrm{R}}\hat{\mathbf{x}}_{\mathrm{R}} = \hat{\mathbf{e}}_1$. Time indices can be added to the frames, if necessary.
$^{\mathrm{R}[k]}_{\mathrm{S}[k']}\mathbf{R}$	The rotation-matrix of $\mathcal{F}_{\mathrm{S}[k']}$ with respect to $\mathcal{F}_{\mathrm{R}[k]}$.
$^{\mathrm{R}}_{\mathrm{S}}\mathbf{t}$	The translation vector $\overrightarrow{\mathcal{O}_{\mathrm{R}}\mathcal{O}_{\mathrm{S}}}$ resolved in $\mathcal{F}_{\mathrm{R}}$.

Transform between two coordinate frames

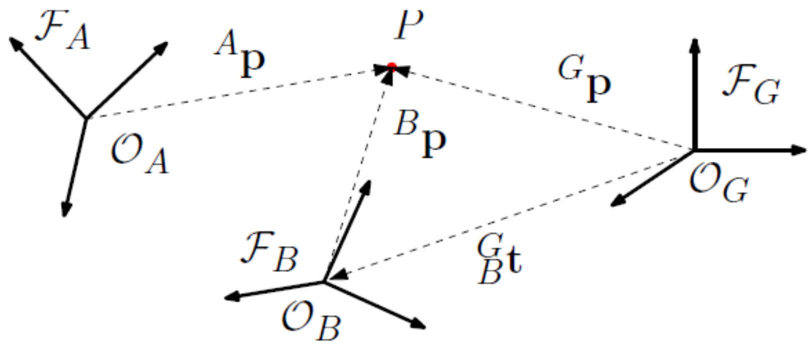
$$^{\mathrm{G}}_{\mathrm{A}}\mathbf{t} \triangleq \overrightarrow{\mathcal{O}_{\mathrm{G}}\mathcal{O}_{\mathrm{A}}} \text{ resolved in } \mathcal{F}_{\mathrm{G}}$$

$$\begin{pmatrix} ^{\mathrm{G}}\mathbf{p} \\ 1 \end{pmatrix} \equiv \begin{pmatrix} ^{\mathrm{G}}_{\mathrm{A}}\mathbf{R} & ^{\mathrm{G}}_{\mathrm{A}}\mathbf{t} \\ \mathbf{0}_{1\times[2,3]} & 1 \end{pmatrix} \begin{pmatrix} ^{\mathrm{A}}\mathbf{p} \\ 1 \end{pmatrix} \qquad ^{\mathrm{G}}_{\mathrm{A}}\mathbf{T} \equiv \left\{ \begin{matrix} ^{\mathrm{G}}_{\mathrm{A}}\mathbf{t} \\ ^{\mathrm{G}}_{\mathrm{A}}\mathbf{R} \end{matrix} \right\}$$

$$\begin{aligned} ^{\mathrm{G}}\mathbf{p} &= {}^{\mathrm{G}}_{\mathrm{A}}\mathbf{R}\ ^{\mathrm{A}}\mathbf{p} + {}^{\mathrm{G}}_{\mathrm{A}}\mathbf{t} \\ &\triangleq {}^{\mathrm{G}}_{\mathrm{A}}\mathbf{T}(\ ^{\mathrm{A}}\mathbf{p}). \end{aligned}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & ^{\mathrm{G}}_{\mathrm{A}}t_x \\ \sin\theta & \cos\theta & ^{\mathrm{G}}_{\mathrm{A}}t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transform: Operations



Transform between two coordinate frames (chaining, compounding):

$$^G_B\mathbf{T} = {}^G_A\mathbf{T}\,{}^A_B\mathbf{T} \equiv \begin{Bmatrix} {}^G_A\mathbf{R}\,{}^A_B\mathbf{t} + {}^G_A\mathbf{t} \\ {}^G_A\mathbf{R}\,{}^A_B\mathbf{R} \end{Bmatrix}$$

Inverse of a Transform :

$$^B_A\mathbf{T} = {}^A_B\mathbf{T}^{-1} \equiv \begin{Bmatrix} -{}^A_B\mathbf{R}^\top\,{}^A_B\mathbf{t} \\ {}^A_B\mathbf{R}^\top \end{Bmatrix}$$

Relative (Difference) Transform :

$$^B_A\mathbf{T} = {}^G_B\mathbf{T}^{-1}\,{}^G_A\mathbf{T}$$

See: **Quick Reference to Geometric Transforms in Robotics** by Kaustubh Pathak on the webpage!

Chaining :

$$ {}^{G}_{R[X+1]}\mathbf{T} = {}^{G}_{R[X]}\mathbf{T}\ {}^{R[X]}_{R[X+1]}\mathbf{T} \equiv \begin{Bmatrix} {}^{G}_{R[X]}\mathrm{R}\ {}^{R[X]}_{R[X+1]}t + {}^{G}_{R[X]}t \\ {}^{G}_{R[X]}\mathrm{R}\ {}^{R[X]}_{R[X+1]}\mathrm{R} \end{Bmatrix} = \begin{Bmatrix} {}^{G}_{R[X+1]}t \\ {}^{G}_{R[X+1]}\mathrm{R} \end{Bmatrix} $$

In 2D Translation:

$$ \begin{bmatrix} {}^{G}_{R[X+1]}t_x \\ {}^{G}_{R[X+1]}t_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos {}^{G}_{R[X]}\theta & -\sin {}^{G}_{R[X]}\theta & {}^{G}_{R[X]}\mathrm{t_x} \\ \sin {}^{G}_{R[X]}\theta & \cos {}^{G}_{R[X]}\theta & {}^{G}_{R[X]}\mathrm{t_y} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^{R[X]}_{R[X+1]}\mathrm{t_x} \\ {}^{R[X]}_{R[X+1]}t_y \\ 1 \end{bmatrix} $$

In 2D Rotation:

$$ {}^{G}_{R[X+1]}\mathrm{R} = \begin{bmatrix} \cos {}^{G}_{R[X+1]}\theta & -\sin {}^{G}_{R[X+1]}\theta \\ \sin {}^{G}_{R[X+1]}\theta & \cos {}^{G}_{R[X+1]}\theta \end{bmatrix} = \begin{bmatrix} \cos {}^{G}_{R[X]}\theta & -\sin {}^{G}_{R[X]}\theta \\ \sin {}^{G}_{R[X]}\theta & \cos {}^{G}_{R[X]}\theta \end{bmatrix} \begin{bmatrix} \cos {}^{R[X]}_{R[X+1]}\theta & -\sin {}^{R[X]}_{R[X+1]}\theta \\ \sin {}^{R[X]}_{R[X+1]}\theta & \cos {}^{R[X]}_{R[X+1]}\theta \end{bmatrix} $$

In 2D Rotation (simple):

$$ {}^{G}_{R[X+1]}\theta = {}^{G}_{R[X]}\theta + {}^{R[X]}_{R[X+1]}\theta $$

In ROS: nav_2d_msgs/Pose2DStamped

- First Message at time 97 : G
- Message at time 103 : X
- Next Message at time 107 : X+1

$$ {}^{G}_{R[X+1]}\mathbf{T} = {}^{G}_{R[X]}\mathbf{T} \; {}^{R[X]}_{R[X+1]}\mathbf{T} $$

```
std_msgs/Header header
  uint32 seq
  time stamp
  string frame_id
geometry_msgs/Pose2D pose2D
  float64 x
  float64 y
  float64 theta
```

$$ {}^{R[X]}_{R[X+1]}t_x $$

$$ {}^{R[X]}_{R[X+1]}t_y $$

$$ {}^{R[X]}_{R[X+1]}\Theta $$

3D Rotation

- Many 3D rotation representations:

https://en.wikipedia.org/wiki/Rotation_formalisms_in_three_dimensions

- Euler angles:
  - Roll: rotation around x-axis
  - Pitch: rotation around y-axis
  - Yaw: rotation around z-axis
  - Apply rotations one after the other…
    - => Order important! E.g.:
      - x-z-x; x-y-z; z-y-x; …
  - ☹ Singularities
  - Gimbal lock in Engineering
    - "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities"