CS289: **Mobile Manipulation Fall 2026**

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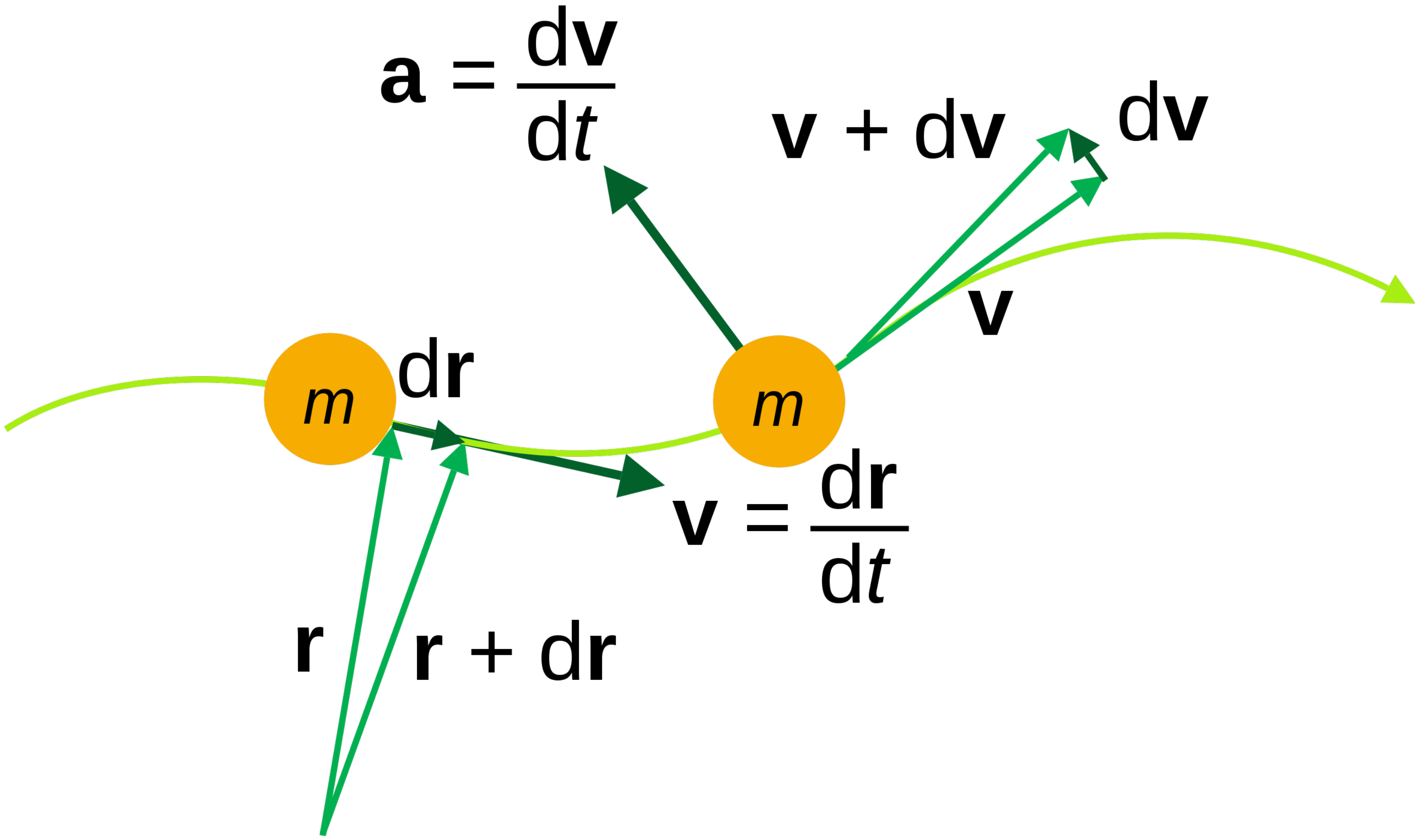
KINEMATICS

What are kinematics?

- Describes the motion of points, bodies (objects), and systems of objects
  - Does not consider the forces that cause them (that would be kinetics)
  - Also known as “the geometry of motion”
- For manipulators
  - Describes the motion of the arm
  - Puts position/ angle and their rate of change (speed) of joints in relation with 3D pose of points on the arm, especially tool center point (tcp, end effector)

What are kinematics?

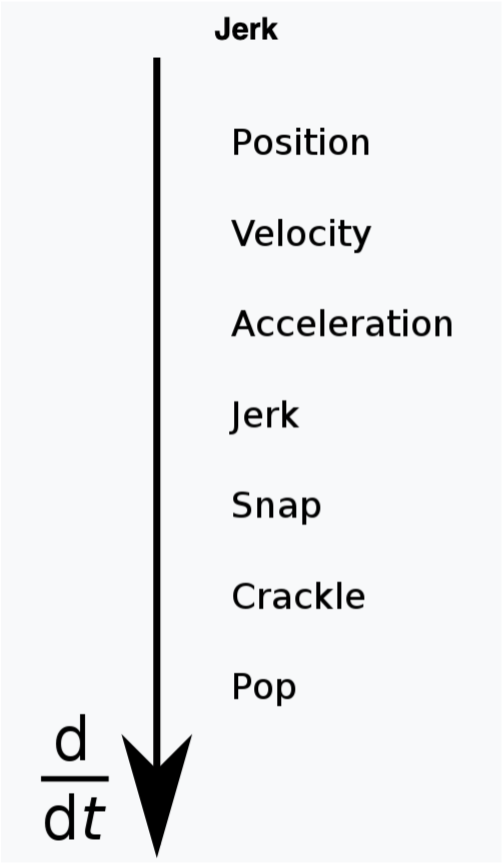


What are kinematics?

- It does not stop at acceleration, but theory involves an arbitrarily high number of derivatives:

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}_0 t^2 + \frac{1}{3}\mathbf{j} t^3 + \ldots$$

$$\boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + \boldsymbol{\omega}_0 t + \frac{1}{2}\boldsymbol{\alpha}_0 t^2 + \frac{1}{3}\zeta t^3 + \ldots$$



Jerk equations: minimal setting for solutions showing chaotic behavior!

In practice

- Often we use finite models to simplify/smoothify the system
  - Locally constant acceleration

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}_0 t^2$$

  - Locally constant velocity

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}_0 t$$

Why do we want to introduce kinematic models

- For control
  - E.g.: Knowledge of how the system is moving is beneficial for reaching the goal pose
- For prediction
  - E.g.: If we have an initial estimate, we can use a kinematic model to generate a prior pose at a later point
- For smoothness
  - E.g.: If we estimate poses, we may constrain their difference to be consistent with some prior or measured velocity
- To impose constraints
  - E.g.: The motion may be more specific and include kinematic constraints

Two types of kinematic constraints

- Holonomic
  - The total number of degrees of freedom is equal to the controllable number of degrees of freedom
  - E.g.: slider on rail (1 DoF)

- Non-holonomic
  - The total number of degrees of freedom is higher than the controllable number of degrees of freedom
  - E.g: A passenger vehicle (3 DoF, while controllable DoF is only 2)

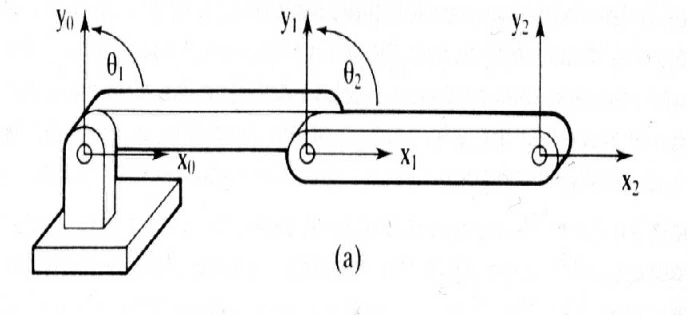
KINEMATIC REPRESENTATIONS

Robot Arm

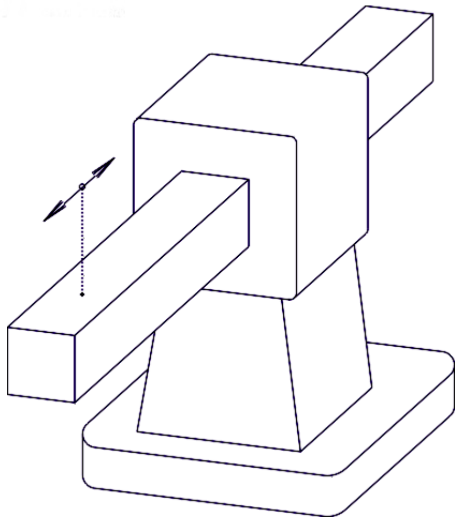
- Consists of Joints and Links …
- and a Base and a Tool (or End-Effector or Tip)

Joints

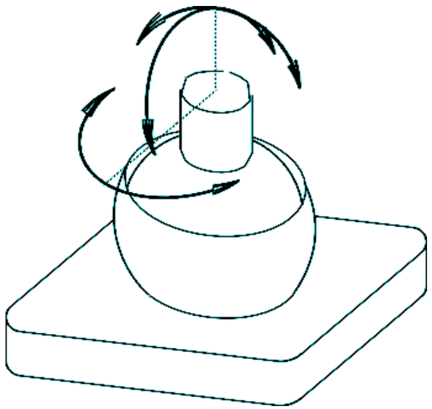
- Revolute Joint: 1DOF



- Prismatic Joint/ Linear Joint: 1DOF



- Spherical Joint: 3DOF

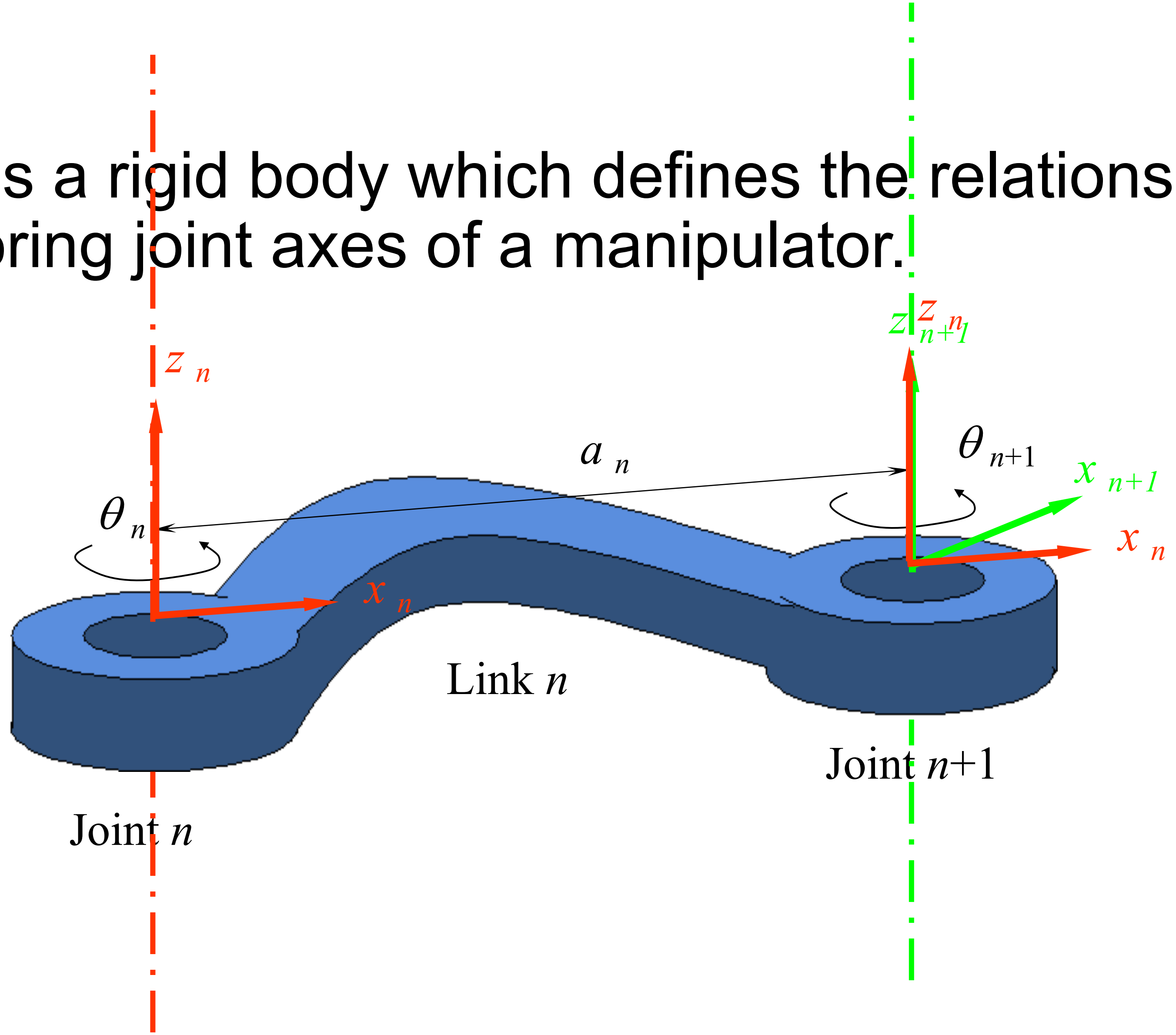


Note on Joints

- Without loss of generality, we will consider only manipulators which have joints with a single degree of freedom.
- A joint having n degrees of freedom can be modeled as n joints of one degree of freedom connected with n-1 links of zero length.
- We could use 6-DoF Transforms to describe their 3D relation – but we can do better (fewer variables):

Link

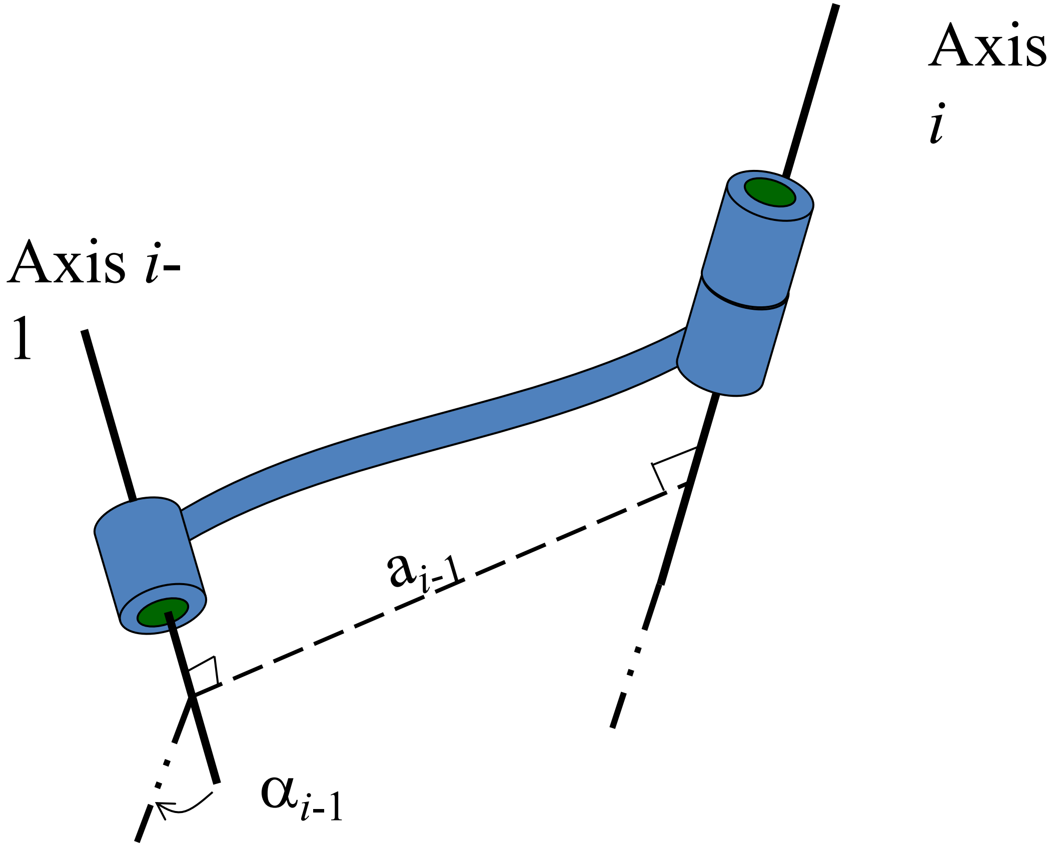
- A link is considered as a rigid body which defines the relationship between two neighboring joint axes of a manipulator.



The Kinematics Function of a Link

- The kinematics function of a link is to maintain a fixed relationship between the two joint axes it supports.

- This relationship can be described with two parameters: the link length a, the link twist α



Link Length

- Is measured along a line which is **mutually perpendicular** to **both** axes.

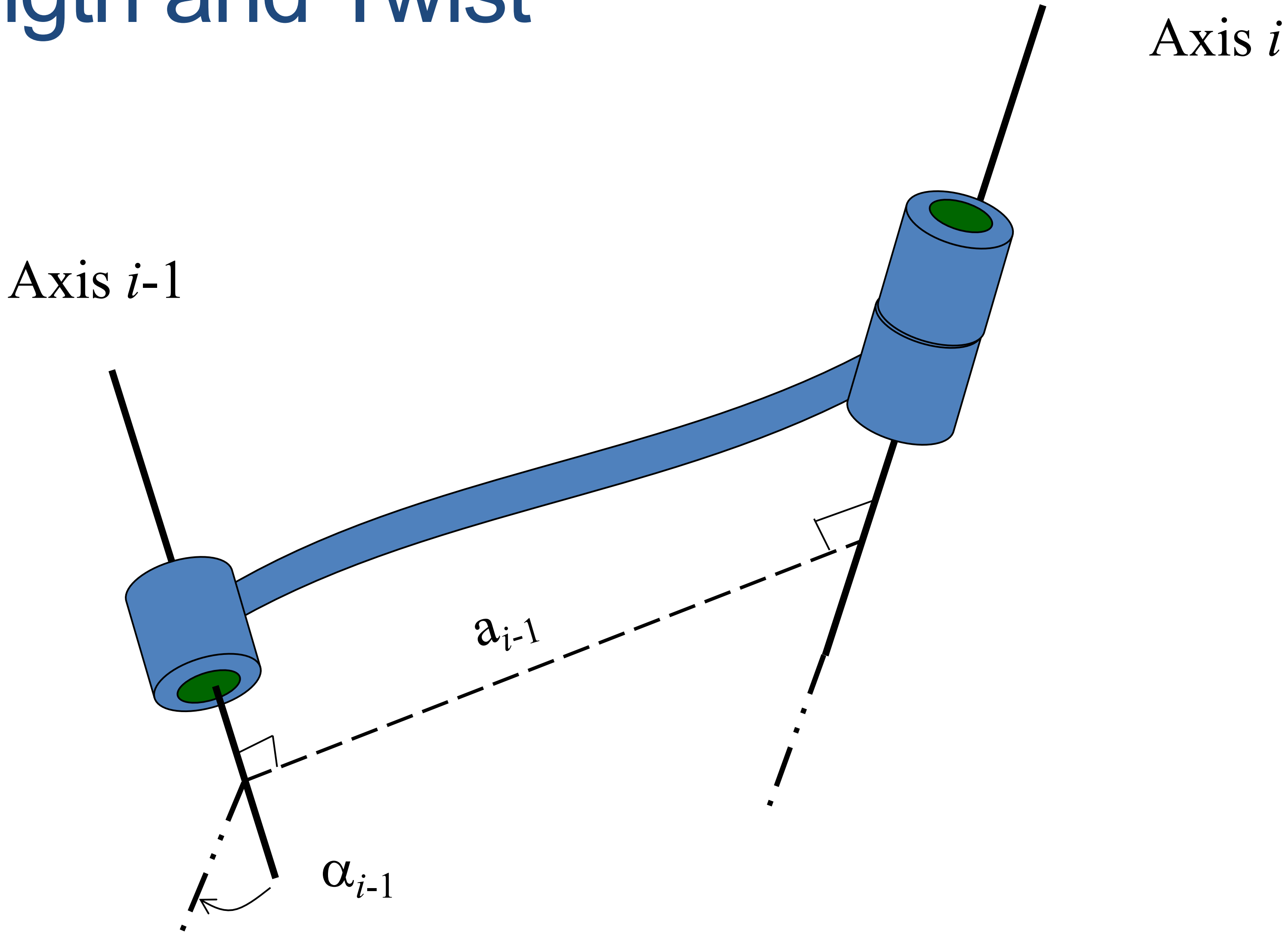
- The mutually perpendicular always exists and is unique except when both axes are parallel.

Link Twist

- Project both axes i-1 and i onto the plane whose normal is the mutually perpendicular line, and measure the angle between them

- Right-hand coordinate system

Link Length and Twist

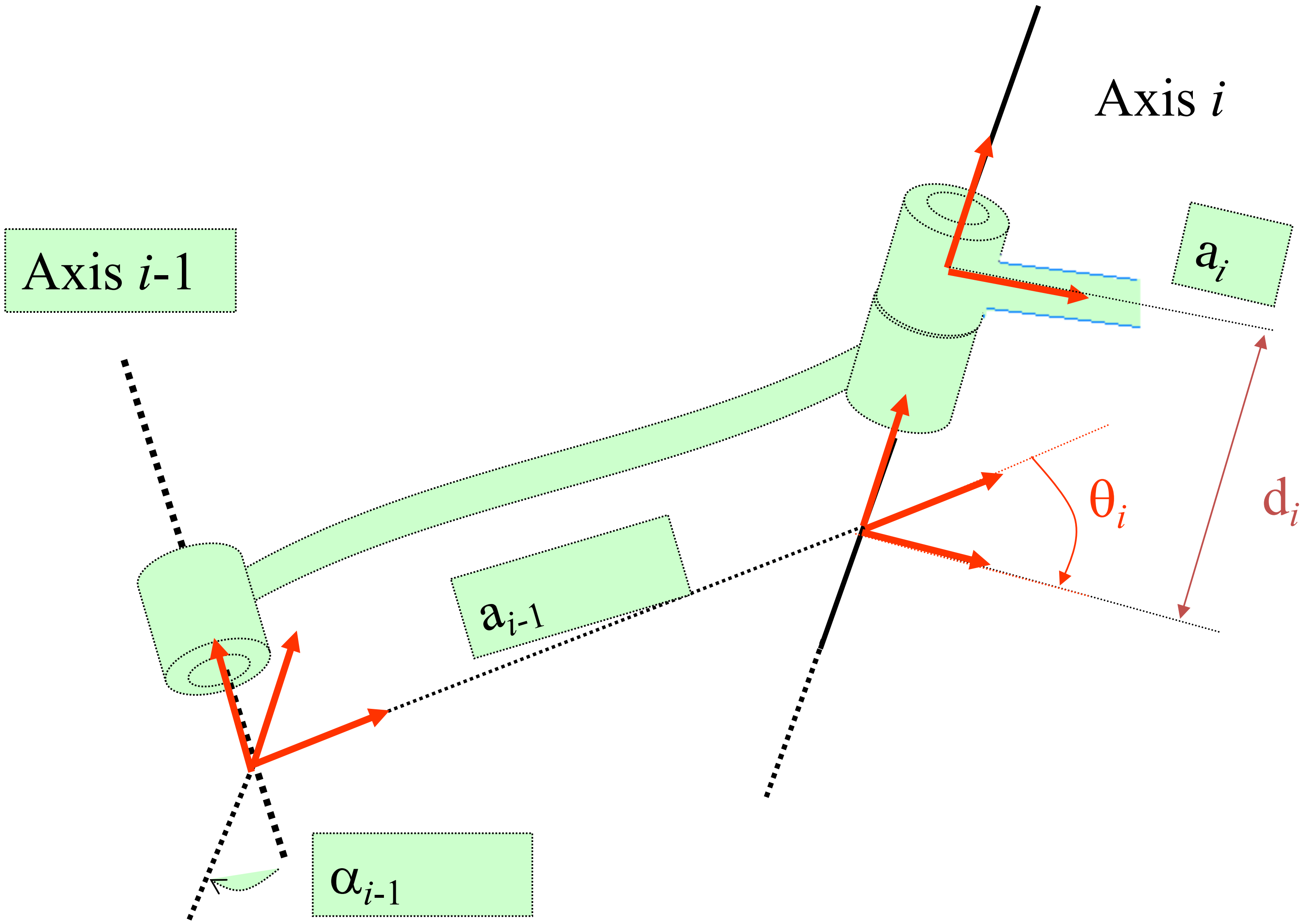


Joint Parameters (the Denavit-Hartenberg (DH) Parameters)

A joint axis is established at the connection of two links. This joint will have two normals connected to it - one for each of the links.

- The relative position of two links is called *link offset* d_n which is the distance between the links (the displacement, along the joint axes between the links).
- The *joint angle* θ_n between the normals is measured in a plane normal to the joint axis.

Link and Joint Parameters



Link and Joint Parameters

4 parameters are associated with each link. You can align the two axis using these parameters.

- Link parameters:

a_n the length of the link.

α_n the twist angle between the joint axes.

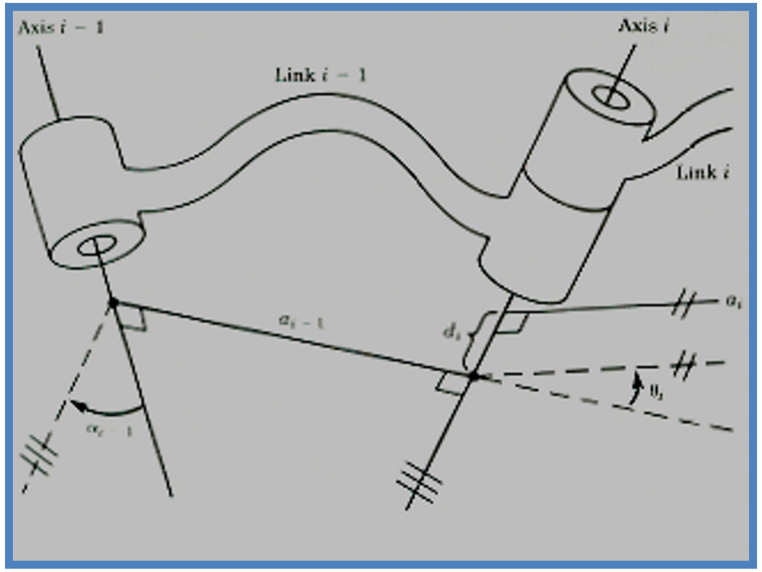
- Joint parameters:

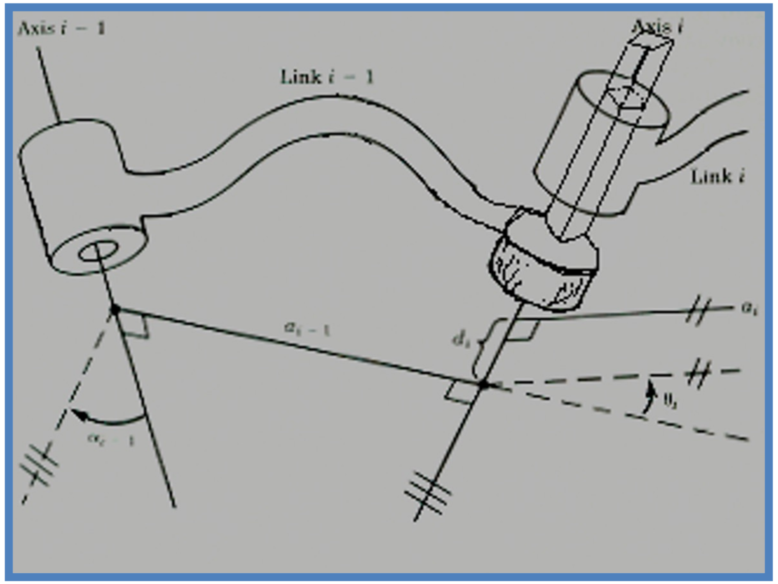
θ_n the angle between the links.

d_n the distance between the links

Link Connection Description:

For Revolute Joints: a, α, and d. are all fixed, then "θ_i" is the. Joint Variable.





For Prismatic Joints: a, α, and θ. are all fixed, then "d_i" is the. Joint Variable.

These four parameters: (Link-Length a_{i-1}), (Link-Twist α_{i-1}), (Link-Offset d_i), (θ_i Joint-Angle θ_i) are known as the *Denavit-Hartenberg Link Parameters*.

Transforms vs. DH Parameters

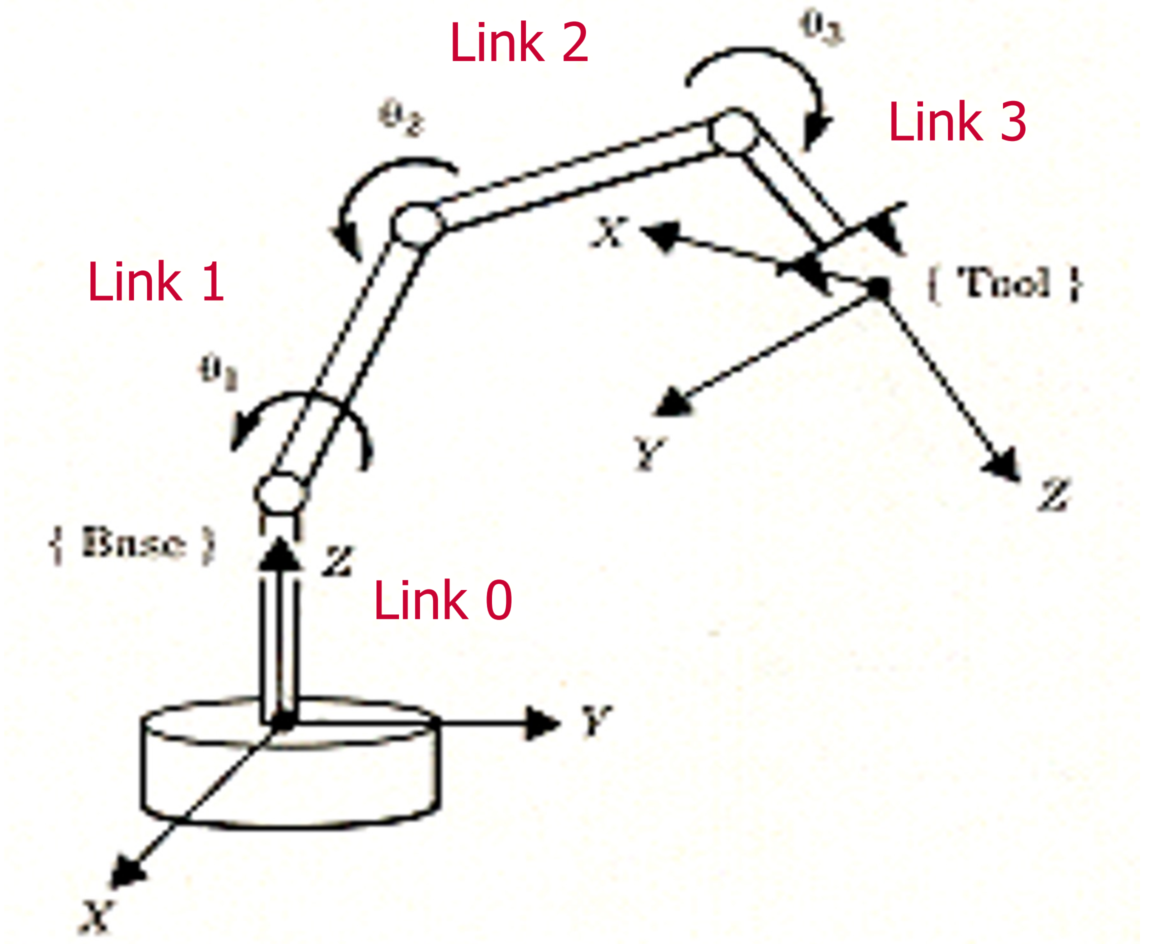
- We could represent an arm using 6 DoF Transforms (ROS MoveIt! is doing this…)

- But:
  General Transform: 6 DoF => DH 4: DoF more efficient reperesentation. E.g. Jacobians for IK will have lots of 0's in DH

- The 2 missing DOFs don't disappear — they're *eliminated by convention*: the DH frame assignment rules fix them, leaving only 4 parameters per link to describe everything you need.
  - E.g. revolute joint: can only rotate around one axis => 2 rotations are not needed!

Links Numbering Convention

Base of the arm:	**Link-0**
1st moving link:	**Link-1**
.	.
.	.
.	.
Last moving link:	**Link-n**



A 3-DOF Manipulator Arm

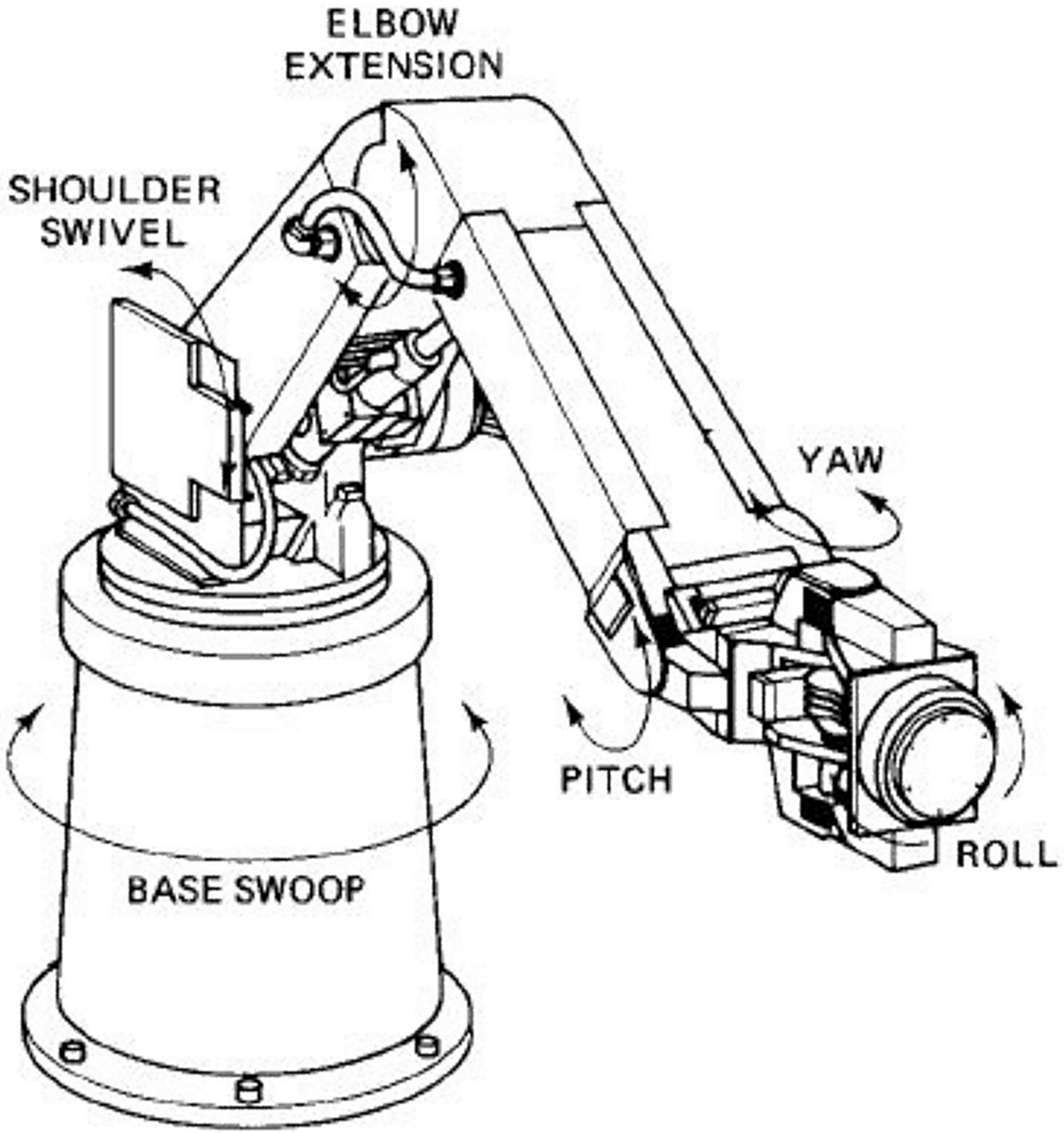
First and Last Links in the Chain

- $a_0 = a_{n.} = 0$
- $\alpha_0 = \alpha_{n.} = 0$
- If joint 1 is revolute: d_0 is fixed and θ_1 is arbitrary
- If joint 1 is prismatic: d_0= arbitrary and θ_1 is fixed

Robot Specifications

Number of axes

- Major axes, (1-3) => position the wrist
- Minor axes, (4-6) => orient the tool
- Redundant, (7-n) => reaching around obstacles, avoiding undesirable configuration



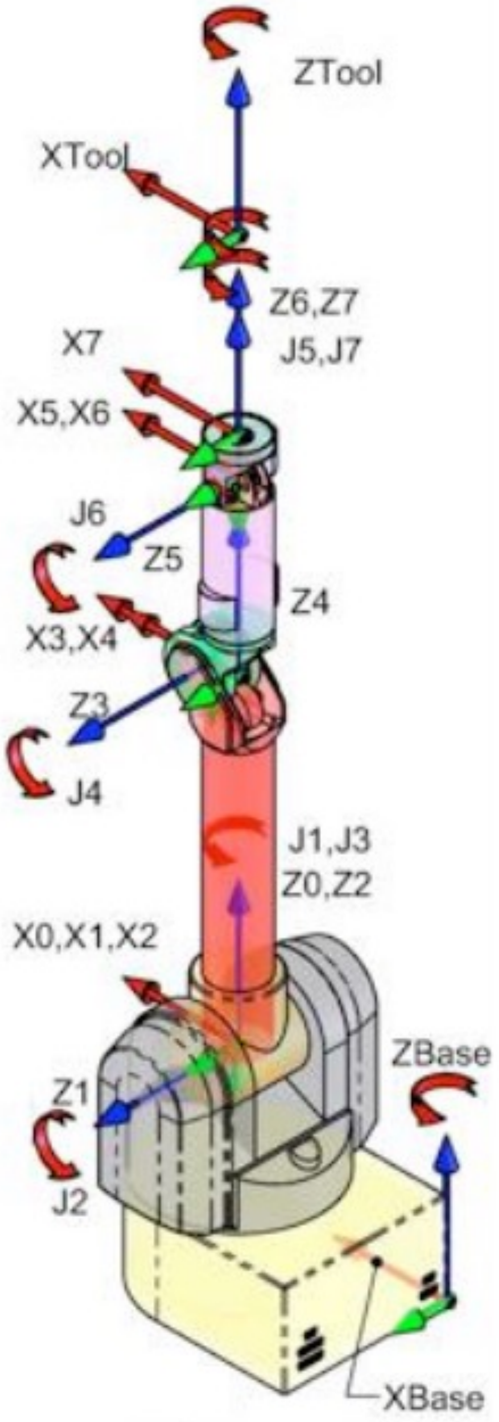
Example: Puma 500

Unimate Puma 500
Oussama Khatib | Used with permission

θ_j	d_j	a_j	α_j
q1	0.0000	0.0000	$\pi/2$
q2	0.0000	0.4318	0
q3	0.1500	0.0203	$-\pi/2$
q4	0.4318	0.0000	$\pi/2$
q5	0.0000	0.0000	$-\pi/2$
q6	0.0000	0.0000	0

Frames

- Choose the base and tool coordinate frame
  - Make your life easy!
- Several conventions
  - Denavit Hartenberg (DH), modified DH, Hayati, etc.



Kinematics

Forward Kinematics (angles to pose) (it is straight-forward -> easy)

What you are given: The constant arm parameters (e.g. DH parameters)
The angle of each joint

What you can find: The pose of any point (i.e. it’s (x, y, z) coordinates & 3D orientation)

Inverse Kinematics (pose to angles) (more difficult)

What you are given: The constant arm parameters (e.g. DH parameters)
The pose of some point on the robot

What you can find: The angles of each joint needed to obtain that pose

Kinematics

Cartesian Space

Tool Frame (E)
(aka End-Effector)
Base Frame (B)

$${}^B_E T = \left\{ \begin{matrix} {}^B_E t \\ {}^B_E R \end{matrix} \right\}$$

Forward Kinematics

$${}^B_E T = f(q)$$

Joint Space

Joint 1 = q_1
Joint 2 = q_2
Joint 3 = q_3
…
Joint n = q_n

$$q = f^{-1}({}^B_E T)$$

Inverse Kinematics

Rigid body transformation
Between coordinate frames

Linear algebra

Kinematics: Velocities

Cartesian Space
Tool Frame (E)
(aka End-Effector)
Base Frame (B)

$$ {}^{B}_{E}V = \begin{Bmatrix} {}^{B}_{E}v \\ {}^{B}_{E}w \end{Bmatrix} $$

v: linear velocity
w: angular velocity

Jacobian

$$ {}^{B}_{E}V = J(q)\dot{q} $$

$$ \dot{q} = J^{-1}(q)\ {}^{B}_{E}V $$

Inverse
Jacobian

Joint Space

Joint 1 = $\dot{q}_1$
Joint 2 = $\dot{q}_2$
Joint 3 = $\dot{q}_3$
...
Joint n = $\dot{q}_n$

Rigid body transformation
Between coordinate frames

Linear algebra

FORWARD KINEMATICS

Forward Kinematics

- DH Parameters to pose:

$$A = R_z(\Theta_j)\, t_z(d_j)\, t_x(a_j)\, R_x(\alpha_j) \qquad \Rightarrow$$

$${}^{B}_{E}T = R_z(\Theta_1)\, t_z(d_1)\, t_x(a_1)\, R_x(\alpha_1) \quad R_z(\Theta_2)\, t_z(d_2)\, t_x(a_2)\, R_x(\alpha_2) \;\cdots\; R_z(\Theta_N)\, t_z(d_N)\, t_x(a_N)\, R_x(\alpha_N)$$

- For revolute joints: only Θ varies – other three are constant for that robot
- For prismatic joints: only d varies – other three are constant for that robot

- For non-holonomic (mobile) robots: Chain the time-varying motion: Odometry

INVERSE KINEMATICS (IK)

Inverse Kinematics (IK)

- Given end effector pose, compute required joint angles
- In simple case, analytic solution exists
  - Use trig, geometry, and algebra to solve
- Possible Problems of Inverse Kinematics
  - Multiple solutions
  - Infinitely many solutions
  - No solutions
  - No closed-form (analytical solution)
- Generally (more DOF) difficult
  - Use Newton’s method

- Analytic solution of 2-link inverse kinematics

$$x^2 + y^2 = a_1^2 + a_2^2 - 2a_1a_2\cos(\pi - \theta_2)$$

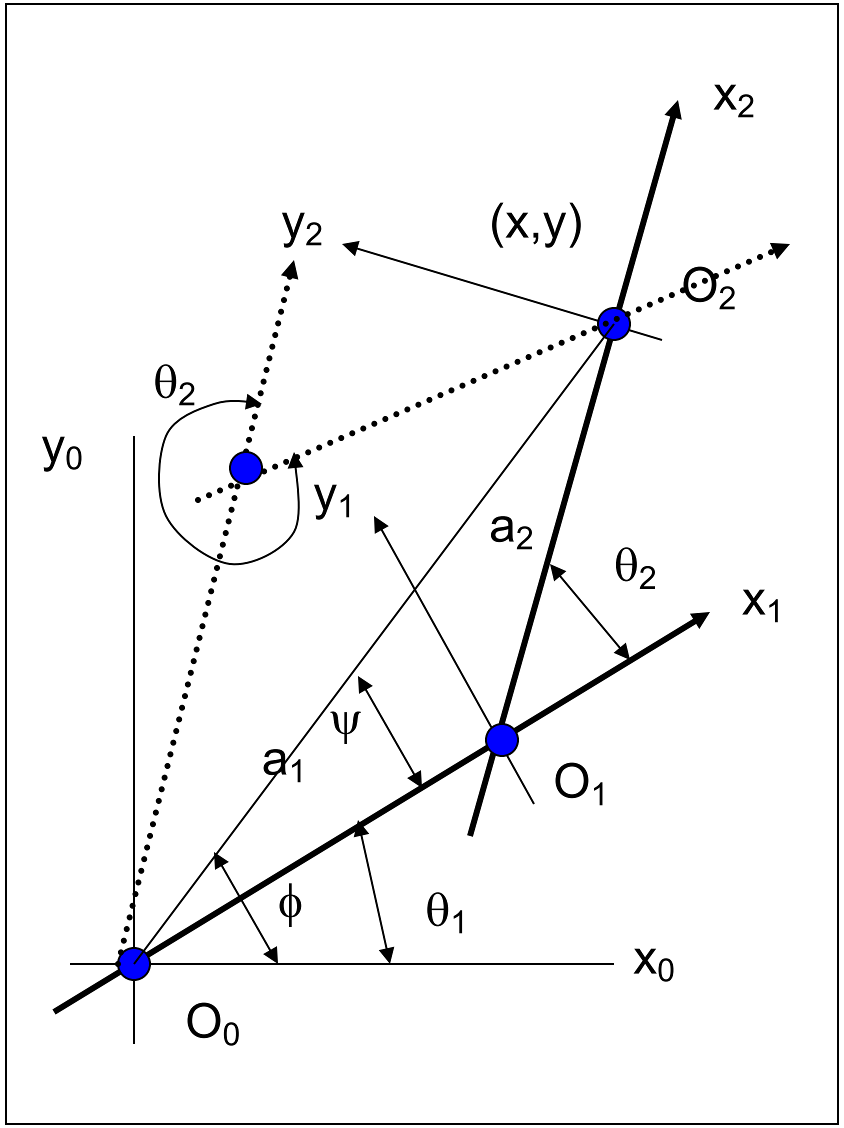
$$\cos\theta_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

for greater accuracy

$$\tan^2\frac{\theta_2}{2} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2a_1a_2 - x^2 - y^2 + a_1^2 + a_2^2}{2a_1a_2 + x^2 + y^2 - a_1^2 - a_2^2}$$

$$= \frac{\left(a_1^2 + a_2^2\right)^2 - \left(x^2 + y^2\right)}{\left(x^2 + y^2\right) - \left(a_1^2 - a_2^2\right)^2}$$

$$\theta_2 = \pm 2\tan^{-1}\sqrt{\frac{\left(a_1^2 + a_2^2\right)^2 - \left(x^2 + y^2\right)}{\left(x^2 + y^2\right) - \left(a_1^2 - a_2^2\right)^2}}$$



- Two solutions: elbow up & elbow down

Iterative IK Solutions

- Often analytic solution is infeasible
- Use Jacobian
- Derivative of function output relative to each of its inputs
- If y is function of three inputs and one output

$$y = f(x_1, x_2, x_3)$$

$$\delta y = \frac{\delta f}{\partial x_1} \cdot \delta x_1 + \frac{\delta f}{\partial x_2} \cdot \delta x_2 + \frac{\delta f}{\partial x_3} \cdot \delta x_3$$

- Represent Jacobian J(X) as a 1x3 matrix of partial derivatives

Jacobian

- In another situation, end effector has 6 DOFs and robotic arm has 6 DOFs
- f(x_1, …, x_6) = (x, y, z, r, p, y)
- Therefore J(X) = 6x6 matrix

$$\begin{bmatrix} \frac{\partial f_x}{\partial x_1} & \frac{\partial f_y}{\partial x_1} & \frac{\partial f_z}{\partial x_1} & \frac{\partial f_r}{\partial x_1} & \frac{\partial f_p}{\partial x_1} & \frac{\partial f_y}{\partial x_1} \\ \frac{\partial f_x}{\partial x_2} & & & & & \\ \frac{\partial f_x}{\partial x_3} & & & & & \\ \frac{\partial f_x}{\partial x_4} & & & & & \\ \frac{\partial f_x}{\partial x_5} & & & & & \\ \frac{\partial f_x}{\partial x_6} & & & & & \end{bmatrix}$$

Jacobian Transpose Method

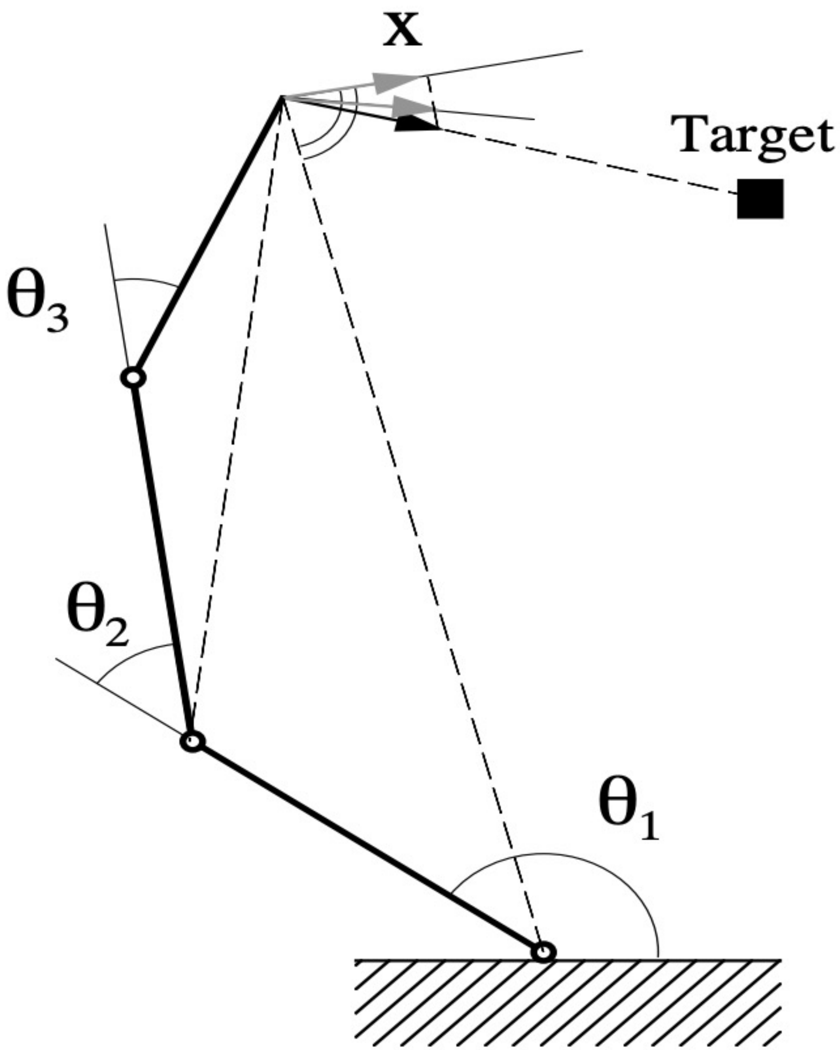
- Relates velocities in parameter space to velocities of outputs

$$\dot{Y} = J(X) \cdot \dot{X}$$

- If we know $Y_{current}$ and $Y_{desired}$, then we subtract to compute $\dot{Y}$
- Invert Jacobian and solve for $\dot{X}$:

$$\dot{X} = \alpha J^T(X) \cdot \Delta Y$$

- Projects difference vector ΔY to those dimensions which can reduce it the most
- Disadvantages:
  - Needs many iterations until convergence in certain configurations (e.g., Jacobian has very small coefficients)
  - Unpredictable joint configurations



Stefan Schaal; USC, Max Planck

Iterative Solution to Inverse Kinematics

- Only holds for high sampling rates or low Cartesian velocities
- “a local solution” that may be “globally” inappropriate
- Problems with singular postures
- Can be used in two ways:
  - As an instantaneous solutions of “which way to take “
  - As an “batch” iteration method to find the correct configuration at a target

ROBOT DESCRIPTION

URDF+Xacro

Unified Robot Description Format (**URDF**) is an XML format for representing a robot model.

It enable to describe kinematic, visual and dynamic properties of a manipulator.

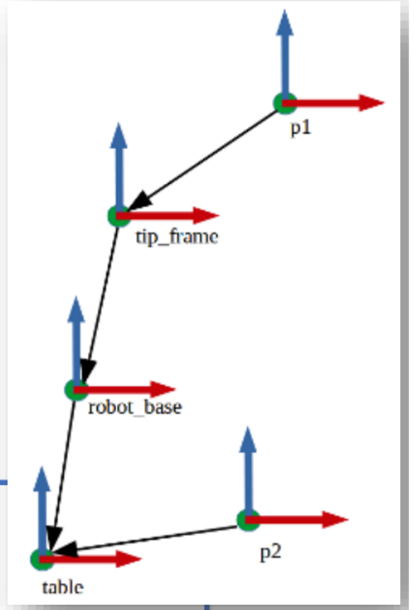
http://wiki.ros.org/urdf

Xacro is an XML macro language: enable construction of shorter and more readable XML files by using macros that expand to larger XML expressions.

http://wiki.ros.org/xacro

ROS provides parsing tools for reading and checking URDF files:

http://wiki.ros.org/urdf/Tutorials



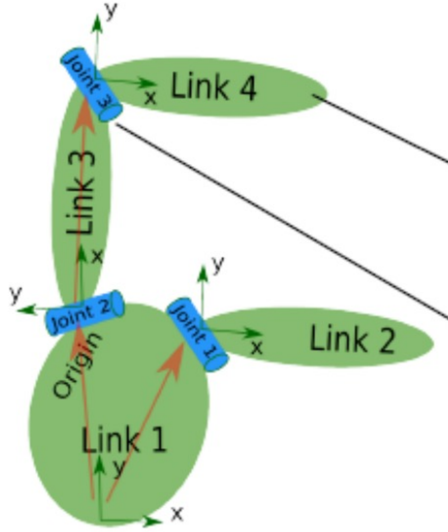

```
<robot name="test_robot">
  <link name="link1" />
  <link name="link2" />
  <link name="link3" />
  <link name="link4" />

  <joint name="joint1" type="continuous">
    <parent link="link1"/>
    <child link="link2"/>
  </joint>
  <joint name="joint2" type="continuous">
    <parent link="link1"/>
    <child link="link3"/>
  </joint>
  <joint name="joint3" type="continuous">
    <parent link="link3"/>
    <child link="link4"/>
  </joint>
</robot>
```

Diego Dall'Alba https://atlas-itn.eu/wp-content/uploads/Presentations/NTA3/univr2_intro_ros_B1.pdf

URDF Simple Example

- Description consists of a set of *link* elements and a set of *joint* elements
- Joints connect the links together



robot.urdf

```
<robot name="robot">
  <link> ... </link>
  <link> ... </link>
  <link> ... </link>

  <joint> .... </joint>
  <joint> .... </joint>
  <joint> .... </joint>
</robot>
```

```
<link name="link_name">
  <visual>
    <geometry>
      <mesh filename="mesh.dae"/>
    </geometry>
  </visual>
  <collision>
    <geometry>
      <cylinder length="0.6" radius="0.2"/>
    </geometry>
  </collision>
  <inertial>
    <mass value="10"/>
    <inertia ixx="0.4" ixy="0.0" …/>
  </inertial>
</link>
```

```
<joint name="joint_name" type="revolute">
  <axis xyz="0 0 1"/>
  <limit effort="1000.0" upper="0.548" … />
  <origin rpy="0 0 0" xyz="0.2 0.01 0"/>
  <parent link="parent_link_name"/>
  <child link="child_link_name"/>
</joint>
```

More info
http://wiki.ros.org/urdf/XML/model

URDF Joint types

1. Fixed: rigid connection
2. Revolute: 1D rotation
3. Continuous: unlimited revolute
4. Prismatic: 1D translation
5. Planar: 2D translation
6. Floating: unlimited 6D

Following material partly from: https://ocw.tudelft.nl/courses/hello-real-world-ros-robot-operating-system/

Standardization in ROS

- ROS Enhancement Proposals (REPs)
  - https://ros.org/reps/rep-0000.html

- REP 103 - *Standard Units of Measure and Coordinate Conventions*
- Right-handed coordinate system
- X+ (forward) and Y+ (left) → Z+ (up)
- SI units:
  - Lengths: meters
  - Angles: radians

Limitations of URDF (1)

1. Robot descriptions cannot be changed
    1. Robot description read only once from parameter server
    2. No standardized way to notify consumers of changes
    3. Risk of *desynchronization* of nodes

2. Only tree structures
    1. Joints have only a single parent and child
    2. Only acyclic, directed graphs (or: trees) can be modeled
    3. Real-world impact: dual-arm manipulation, parallel grippers, etc

Limitations of URDF (2)

3. No `<sensor></sensor>`
    1. Sensor meta-data cannot be incorporated into URDF directly
    2. Alternatives exist, but data is distributed
    3. Diminishes value of URDF as *central* robot description

4. Low reusability of URDFs
    1. Only a single `<robot>` tag in a URDF
    2. No support for import of remote files
    3. Composite scenes have to be merged manually
    4. No way to compose multiple URDFs

Solution: XACRO (XML Macros)

- Programmatic URDF generation
- Templates
- Parameters

arm_macro.xacro

```
<xacro:macro name="arm" params="parent arm_name">
  <link name="${arm_name}_link_1" />
  <joint name="${arm_name}_joint_1" type="..">
    <parent link="${parent}" />
    <child link="${arm_name}_link_1" />
  </joint>
</xacro:macro>
```

robot.xacro

```
[..]
  <link name="torso" />
[..]
  <xacro:arm parent="torso" arm_name="left" />
  <xacro:arm parent="torso" arm_name="right" />
[..]
```

XACRO

- Import macros from other files
- Parameterize templates
- Composite robots & scenes easier:
    - Import macro from file
    - Invoke macro

- Disadvantage: XACRO not directly compatible with URDF
- Transformation needed
- Command:

```
$ rosrun xacro xacro /path/to/robot.xacro > robot.urdf
```

- Also checks for a valid XACRO file

Validate URDFs

- Validate URDFs: check_urdf
- checks syntax (ie: legal combinations of URDF keywords)
- not semantics (ie: meaning or real-world correctness)
- Command:

```
$ check_urdf tiny_robot.urdf
```

```
Error: Failed to build tree:
  child link [link_3] of joint [joint_1] not found
ERROR: Model Parsing the xml failed
```

tiny_robot.urdf

```
<robot name="tiny_robot">
  <link name="link_1" />
  <link name="link_2" />
  <joint name="joint_1" type="continuous">
    <parent link="link_1" />
    <child link="link_3" />
  </joint>
</robot>
```

2. state_publisher node

2.1 ROS API

2.1.1 Subscribed topics

`joint_states` (sensor_msgs/JointState)

joint position information

2.1.2 Parameters

`robot_description` (`urdf map`)

The urdf xml robot description. This is accessed via `urdf_model::initParam`

`tf_prefix` (`string`)

Set the tf prefix for namespace-aware publishing of transforms. See tf_prefix for more details.

`publish_frequency` (`double`)

Publish frequency of state publisher, default: 50Hz.

`ignore_timestamp` (`bool`)

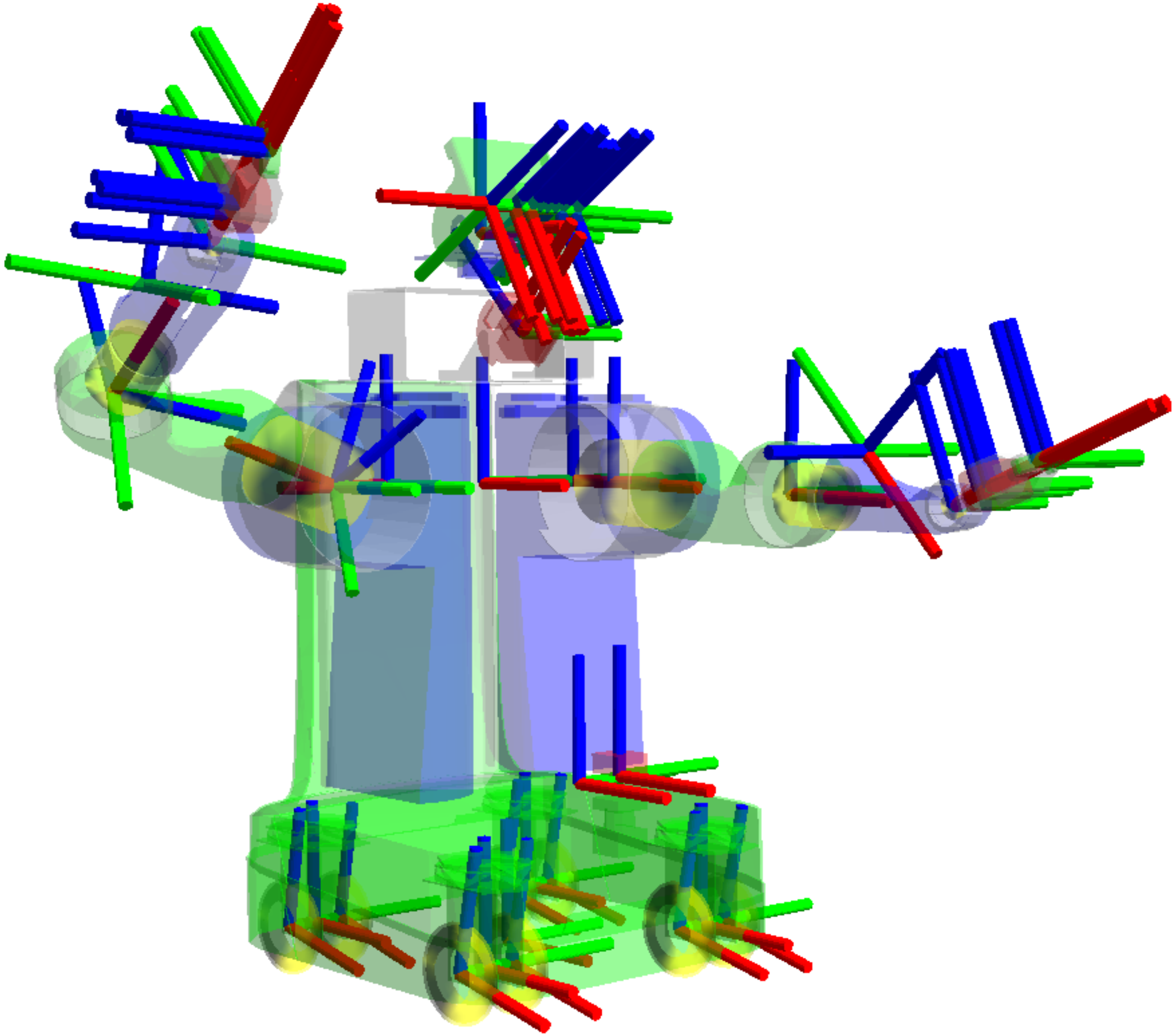
If true, ignore the publish_frequency and the timestamp of joint_states and publish a tf for each of the received joint_states. Default is "false".

`use_tf_static` (`bool`)

Set whether to use the /tf_static latched static transform broadcaster. Default: true.

ROS: 3D Transforms : TF

- http://wiki.ros.org/tf
- http://wiki.ros.org/tf/Tutorials



PICK & PLACE