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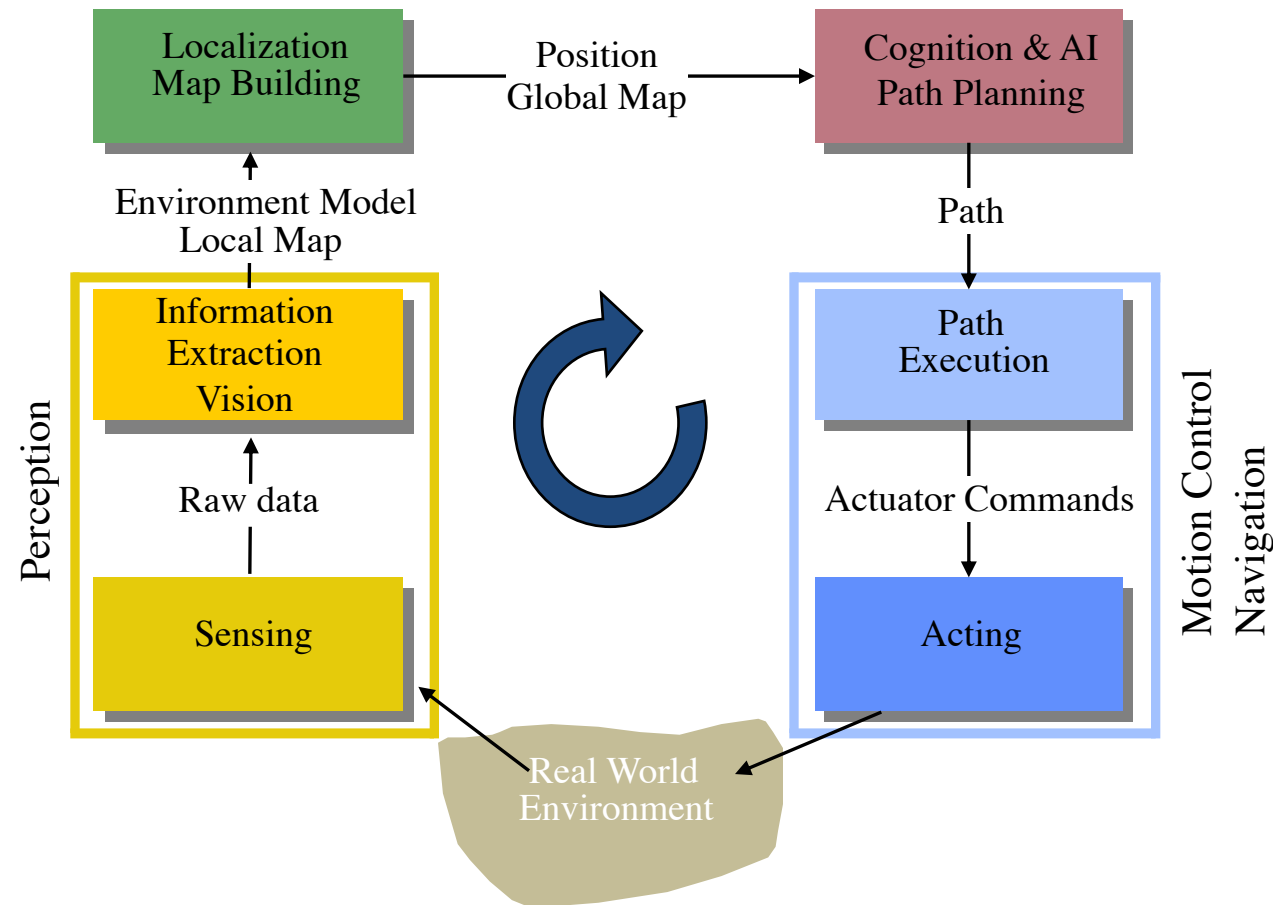
CS283: Robotics Fall 2020: Kinematics

Sören Schwertfeger / 师泽仁

ShanghaiTech University

KINEMATICS

General Control Scheme for Mobile Robot Systems



Motivation

- Autonomous mobile robots move around in the environment.

Therefore **ALL** of them:

- They need to know **where** they **are**.
- They need to know **where** their **goal** is.
- They need to know **how** to get there.

- **Odometry!**

- Robot:

- I know how fast the wheels turned =>
- I know how the robot moved =>
- I know where I am 😊

Odometry

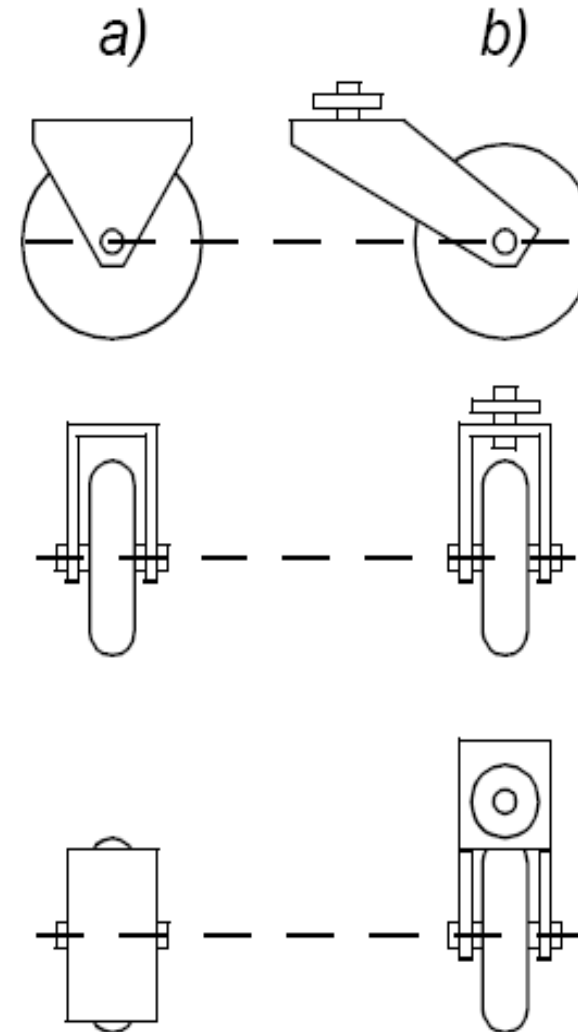
- Robot:
 - I know how fast the wheels turned =>
 - I know how the robot moved =>
 - I know where I am 😊
- Marine Navigation: Dead reckoning (using heading sensor)
- Sources of error (AMR pages 269 - 270):
 - Wheel slip
 - Uneven floor contact (non-planar surface)
 - Robot kinematic: tracked vehicles, 4 wheel differential drive..
 - Integration from speed to position: Limited resolution (time and measurement)
 - Wheel misalignment
 - Wheel diameter uncertainty
 - Variation in contact point of wheel

Mobile Robots with Wheels

- Wheels are the most appropriate solution for most applications
- Three wheels are sufficient to guarantee stability
- With more than three wheels an appropriate suspension is required
- Selection of wheels depends on the application

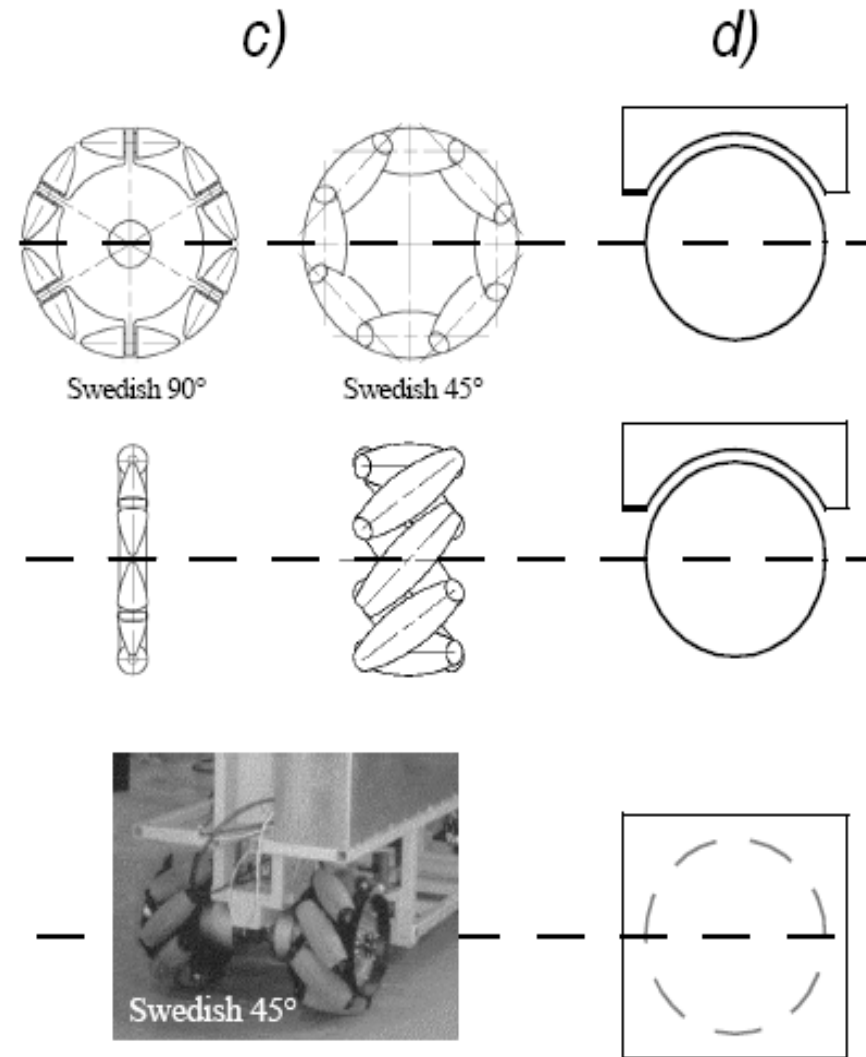
The Four Basic Wheels Types

- a) Standard wheel: Two degrees of freedom; rotation around the (motorized) wheel axle and the contact point
- b) Castor wheel: Three degrees of freedom; rotation around the wheel axle, the contact point and the castor axle



The Four Basic Wheels Types

- c) Swedish wheel: Three degrees of freedom; rotation around the (motorized) wheel axle, around the rollers and around the contact point
- d) Ball or spherical wheel: Suspension technically not solved

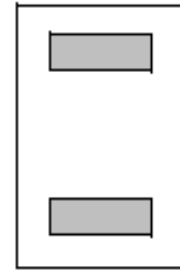


Characteristics of Wheeled Robots and Vehicles

- Stability of a vehicle is be guaranteed with 3 wheels
 - center of gravity is within the triangle with is formed by the ground contact point of the wheels.
- Stability is improved by 4 and more wheel
 - however, this arrangements are hyperstatic and require a flexible suspension system.
- Bigger wheels allow to overcome higher obstacles
 - but they require higher torque or reductions in the gear box.
- Most arrangements are non-holonomic (see chapter 3)
 - require high control effort
- Combining actuation and steering on one wheel makes the design complex and adds additional errors for odometry.

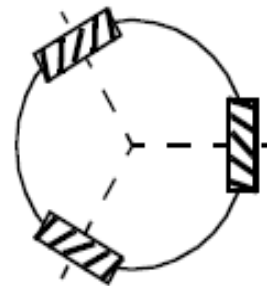
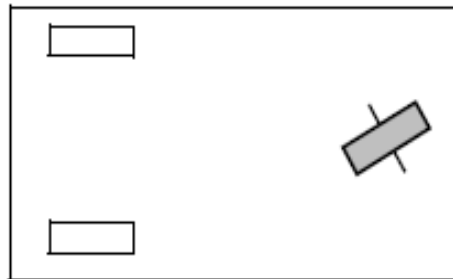
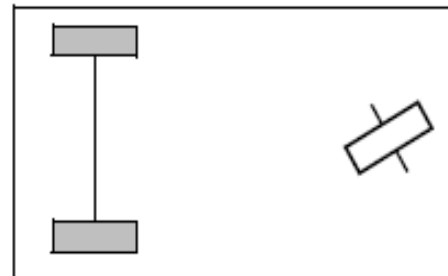
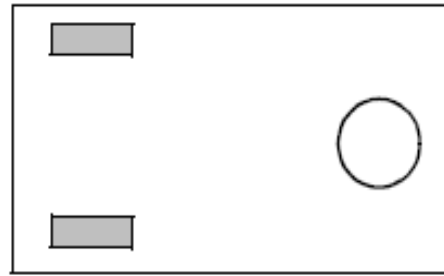
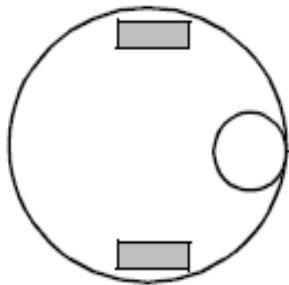
Different Arrangements of Wheels I

- Two wheels

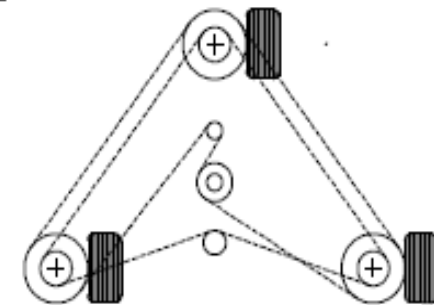


Center of gravity below axle

- Three wheels



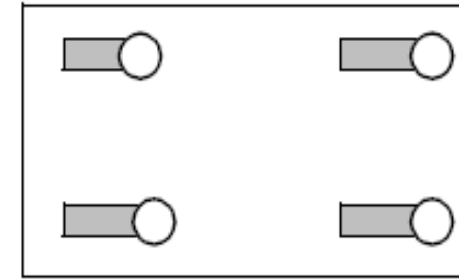
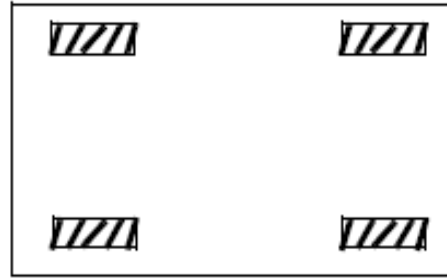
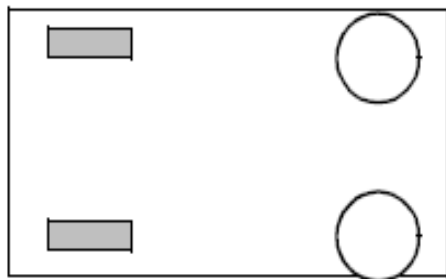
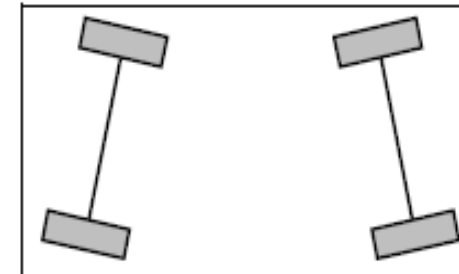
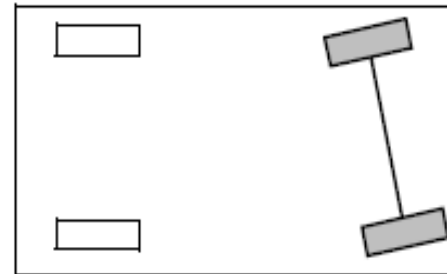
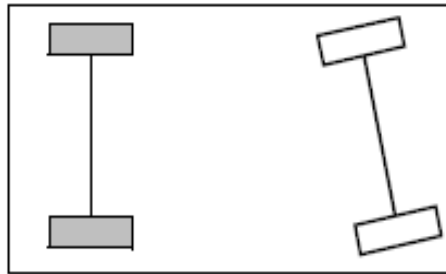
Omnidirectional Drive



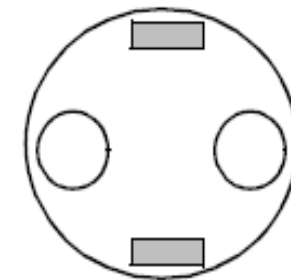
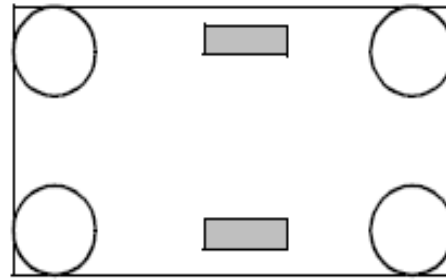
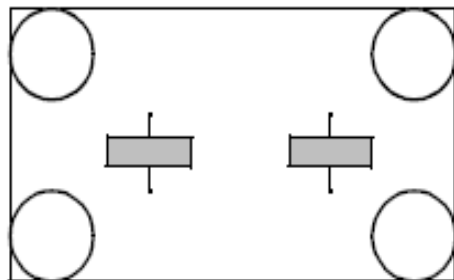
Synchro Drive

Different Arrangements of Wheels II

- Four wheels

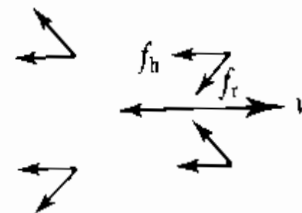
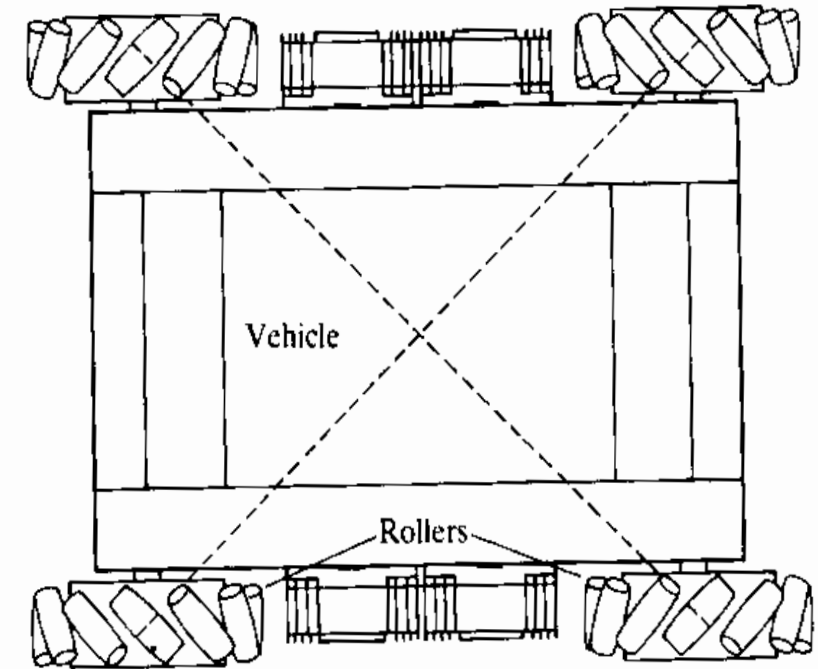
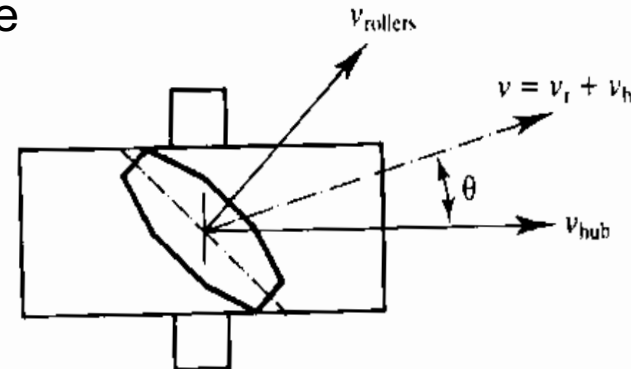
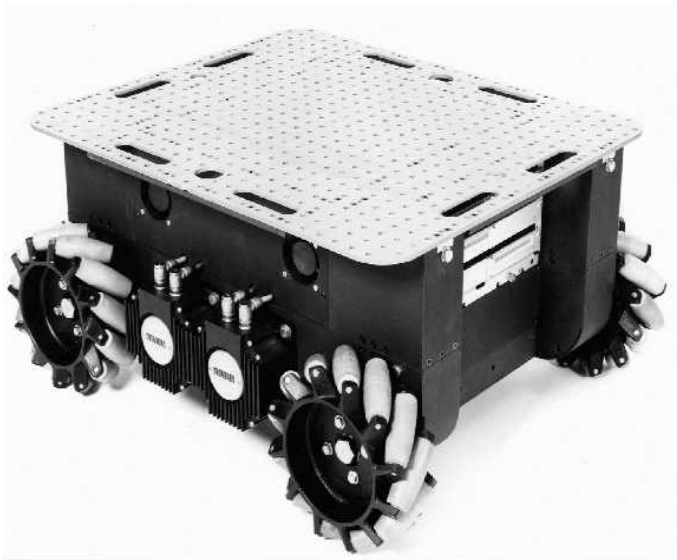


- Six wheels

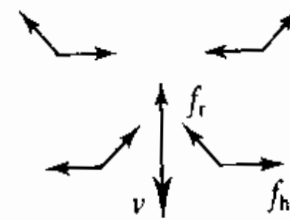


Uranus, CMU: Omnidirectional Drive with 4 Wheels

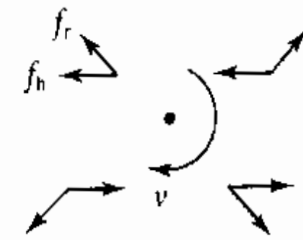
- Movement in the plane has 3 DOF
 - thus only three wheels can be independently controlled
 - It might be better to arrange three swedish wheels in a triangle



Forward



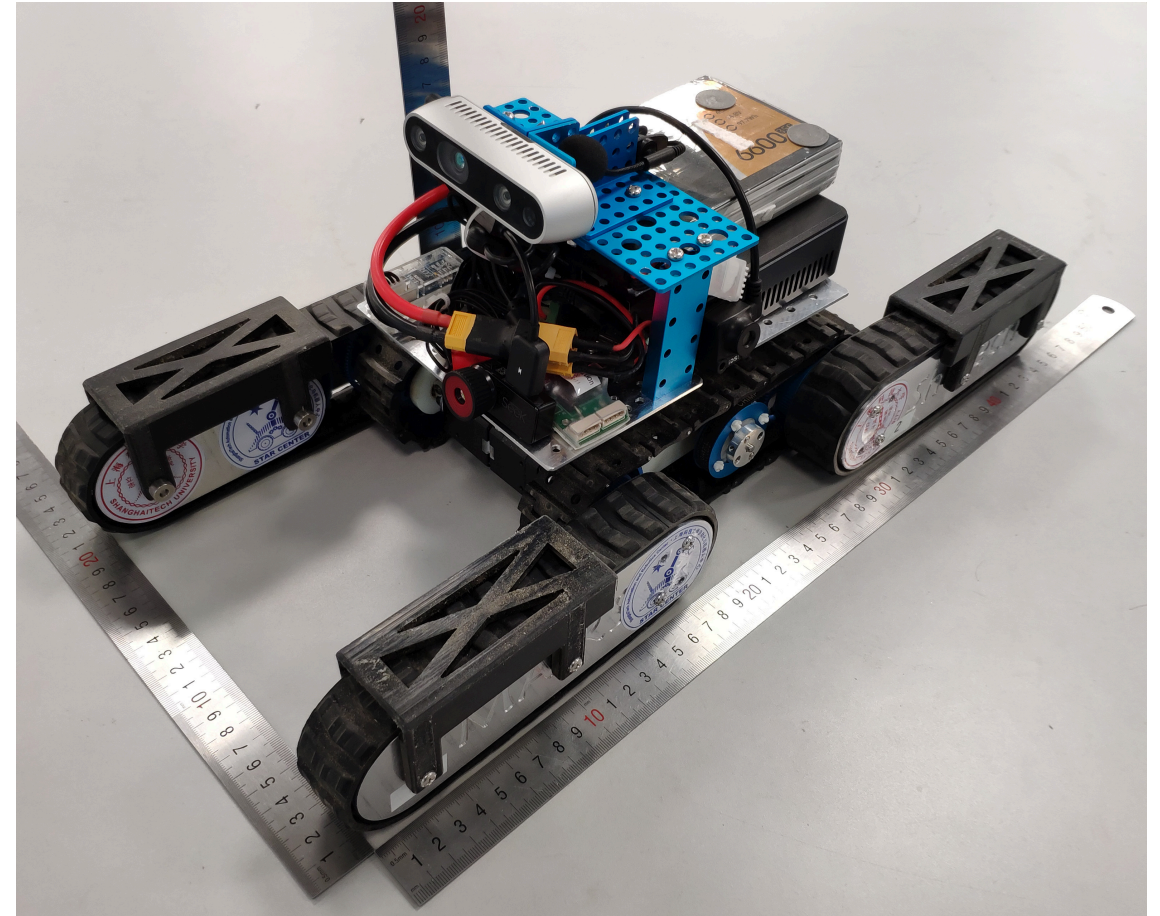
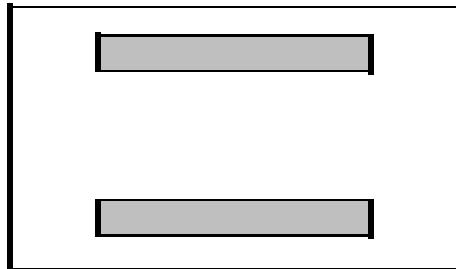
Right



Clockwise

MARS Rescue Robot: Tracked Differential Drive

- Kinematic Simplification:
 - 2 Wheels, located at the center

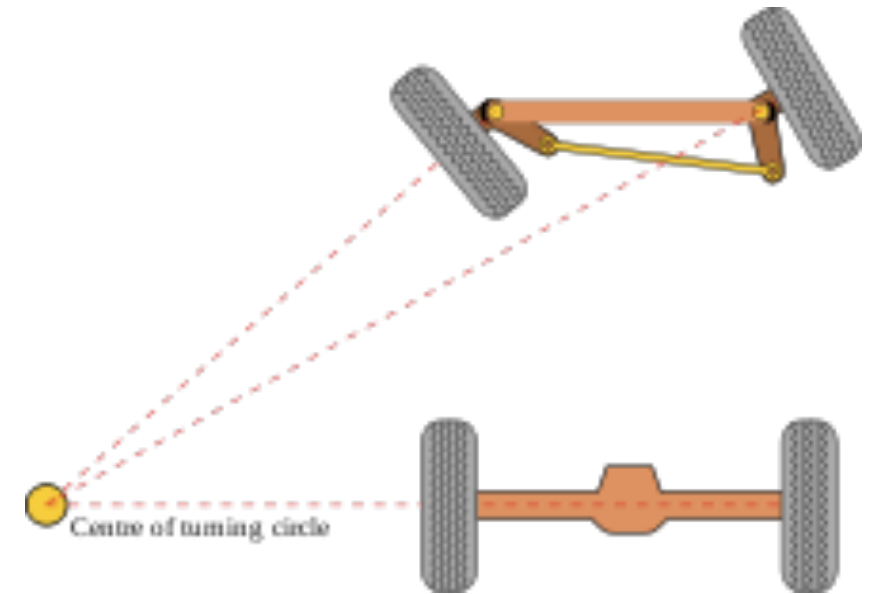


Differential Drive Robots



Ackermann Robot

- No sideways slip than differential drive during turning 😊
- Cannot turn on the spot 😞



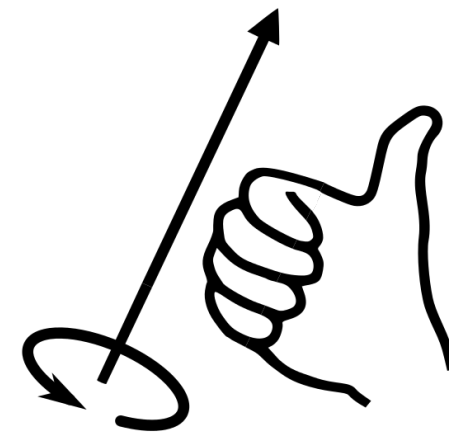
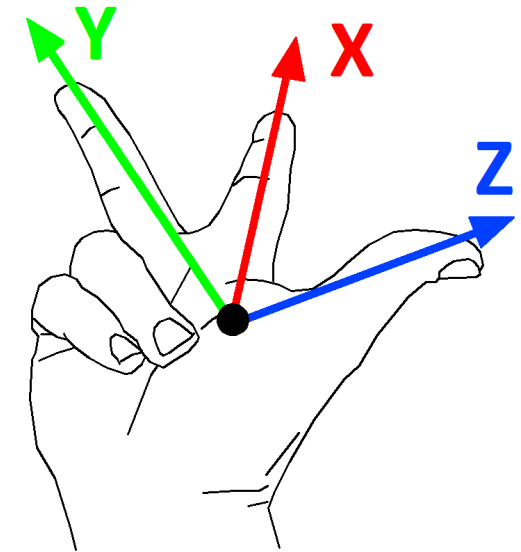
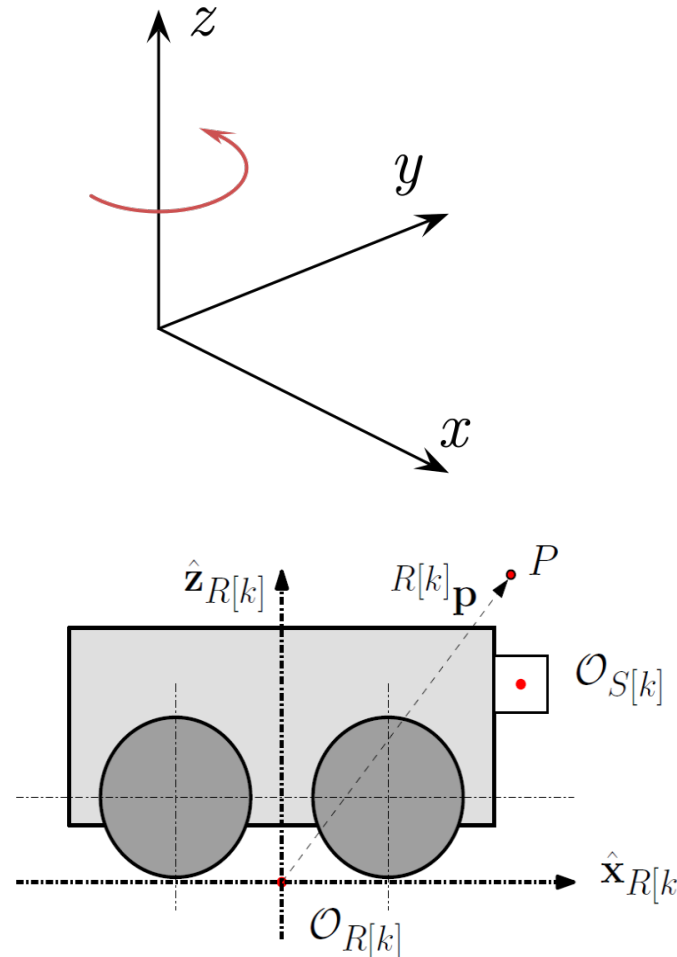
Introduction: Mobile Robot Kinematics

- Aim
 - Description of mechanical behavior of the robot for *design* and *control*
 - Similar to robot manipulator kinematics
 - However, mobile robots can move unbound with respect to its environment
 - there is no direct way to measure the robot's position
 - Position must be integrated over time
 - Leads to inaccuracies of the position (motion) estimate
 - > *the number 1 challenge in mobile robotics*

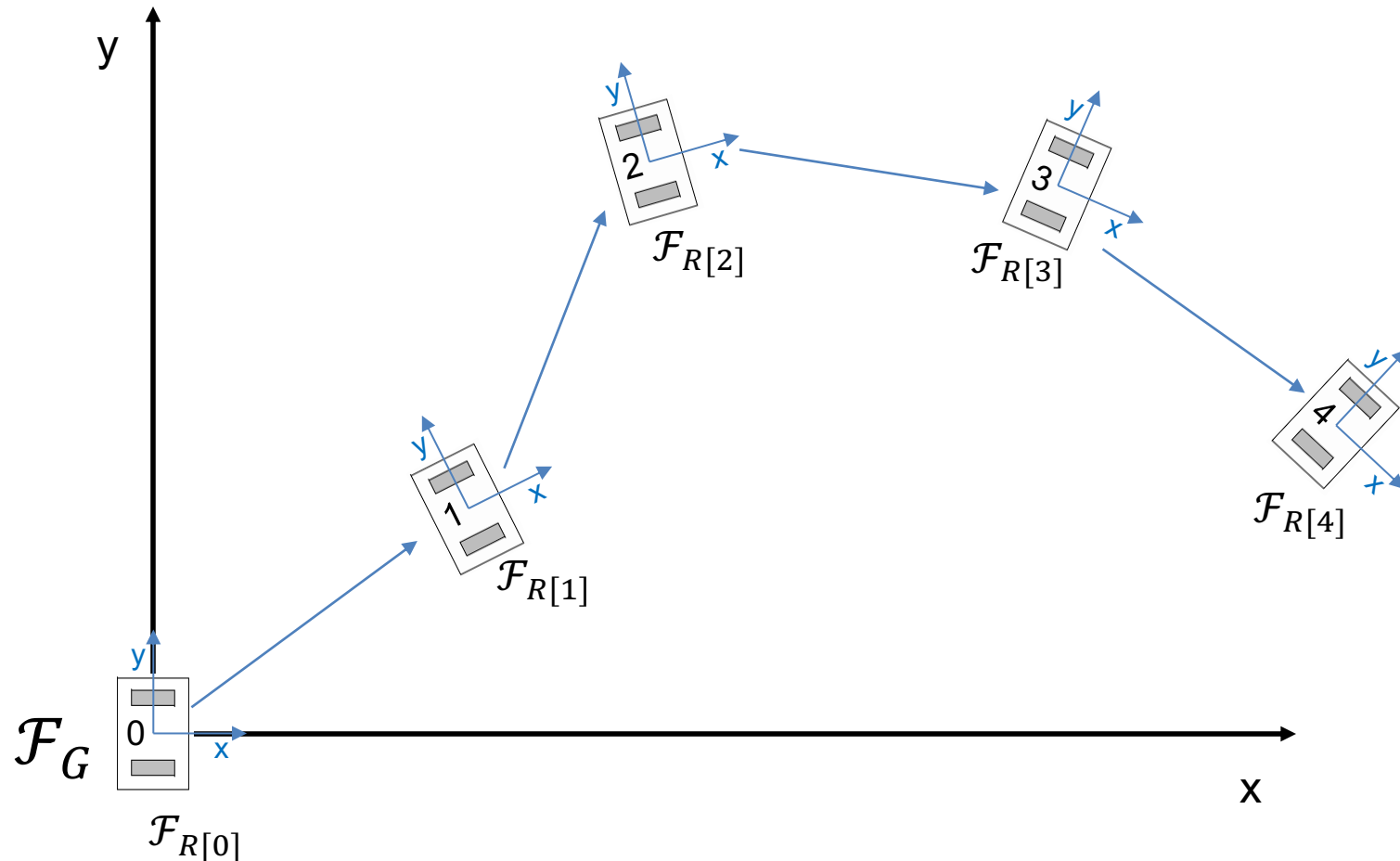
COORDINATE SYSTEM

Right Hand Coordinate System

- Standard in Robotics
- Positive rotation around X is anti-clockwise
- Right-hand rule mnemonic:
 - Thumb: z-axis
 - Index finger: x-axis
 - Second finger: y-axis
 - Rotation: Thumb = rotation axis, positive rotation in finger direction
- Robot Coordinate System:
 - X front
 - Z up (Underwater: Z down)
 - Y ???



Odometry



With respect to the robot start pose:

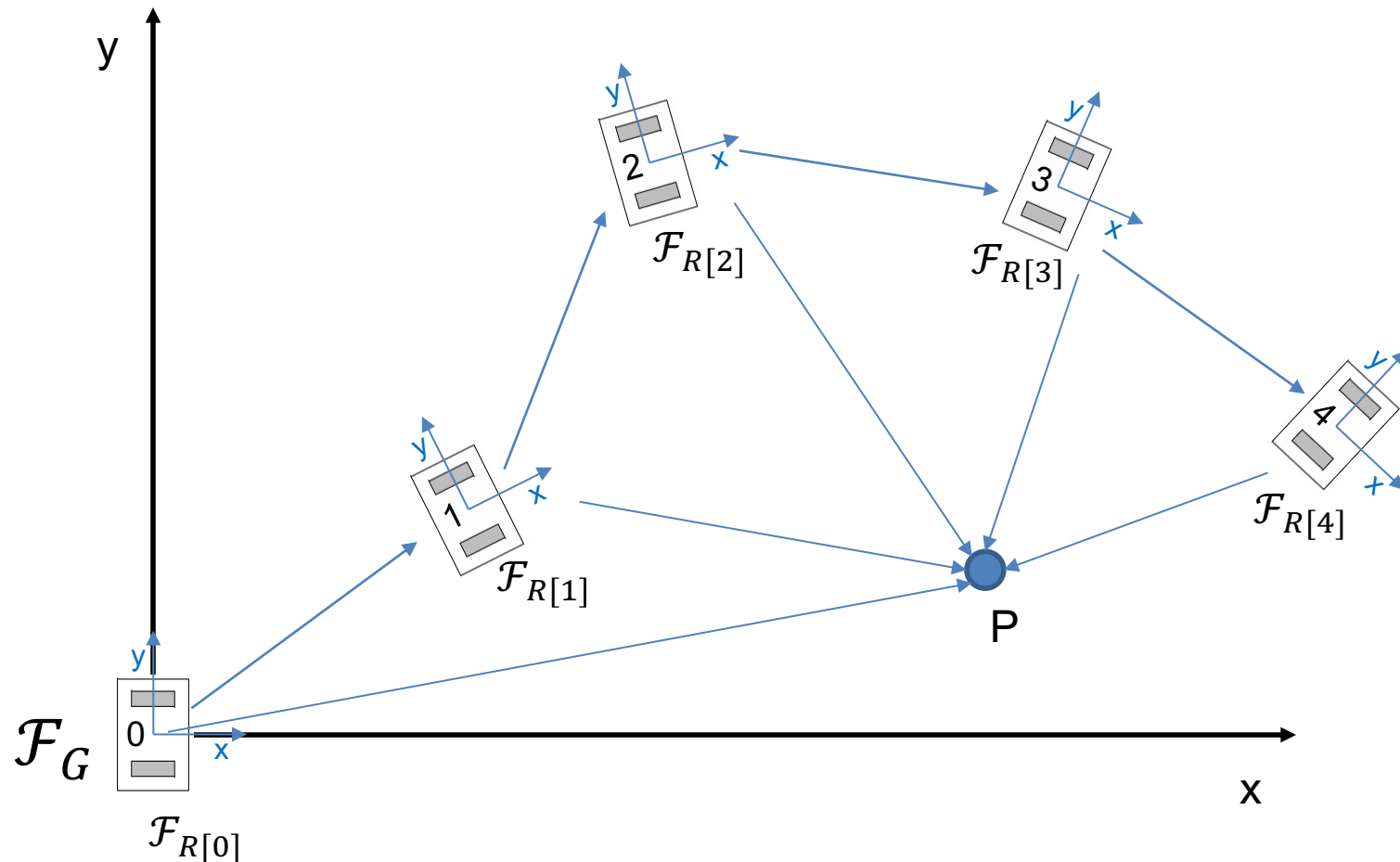
Where is the robot now?

Two approaches – same result:

- Geometry (easy in 2D)
- Transforms (better for 3D)

$\mathcal{F}_{R[X]}$: The **F**rame of reference (the local coordinate system) of the **R**obot at the time **X**

Use of robot frames $\mathcal{F}_{R[X]}$

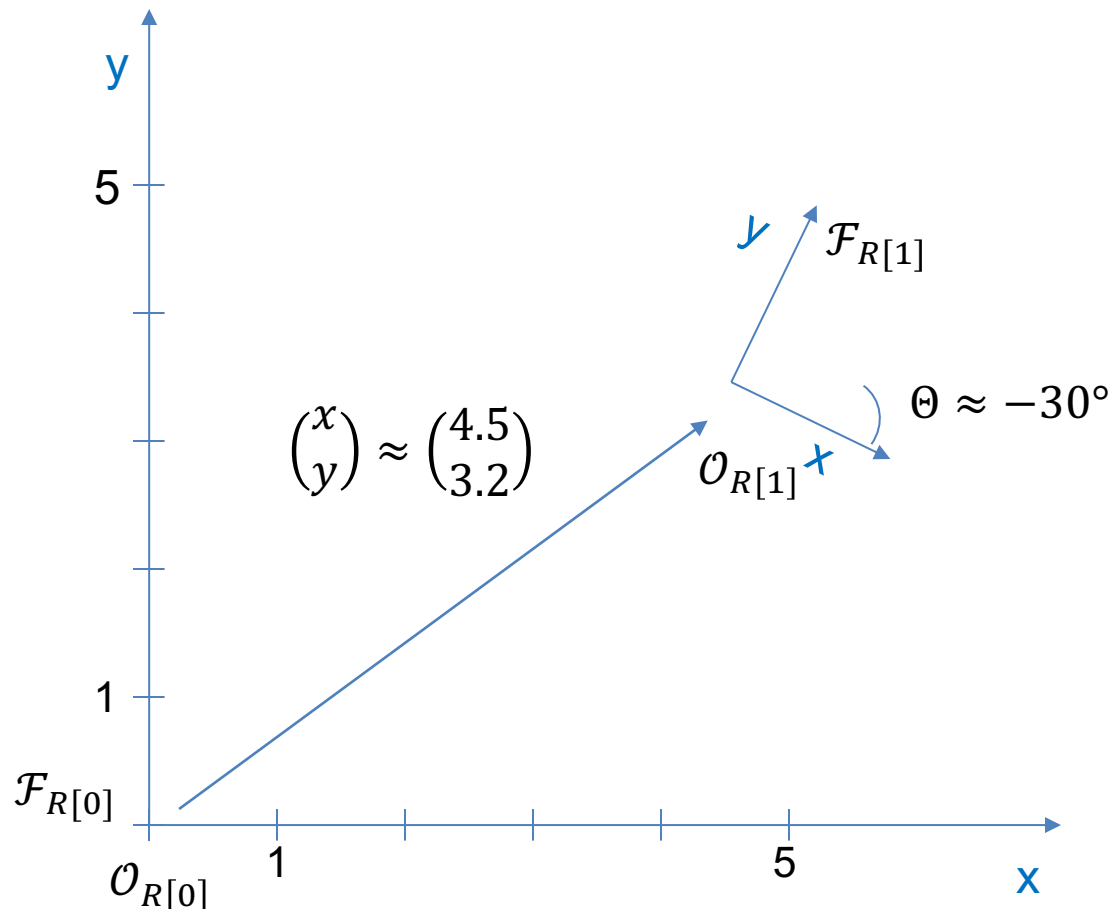


$\mathcal{O}_{R[X]}$: Origin of $\mathcal{F}_{R[X]}$
(coordinates (0, 0))

$\overrightarrow{\mathcal{O}_{R[X]}P}$: position vector from $\mathcal{O}_{R[X]}$ to point P - $\begin{pmatrix} x \\ y \end{pmatrix}$

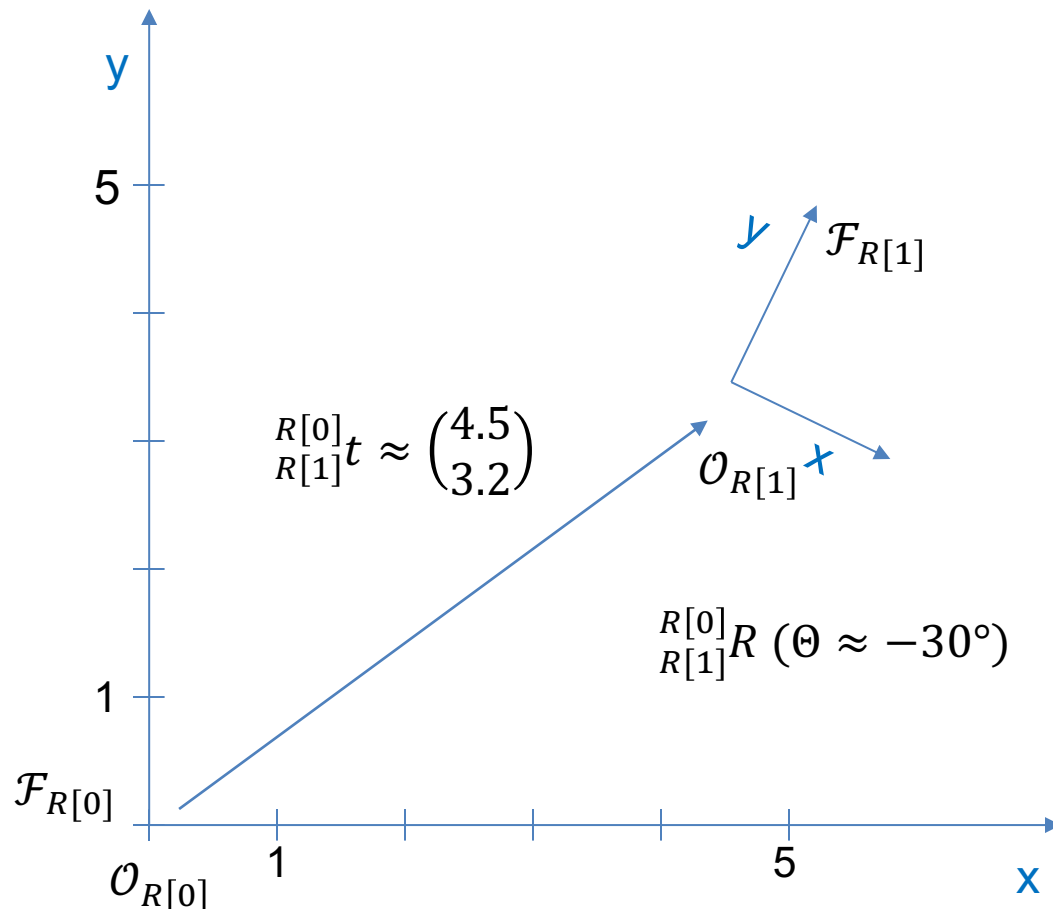
- Object P is observed at times 0 to 4
- Object P is static (does not move)
- The Robot moves (e.g. $\mathcal{F}_{R[0]} \neq \mathcal{F}_{R[1]}$)
- \Rightarrow (x, y) coordinates of P are different in all frames, for example:
 - $\overrightarrow{\mathcal{O}_{R[0]}P} \neq \overrightarrow{\mathcal{O}_{R[1]}P}$

Position, Orientation & Pose



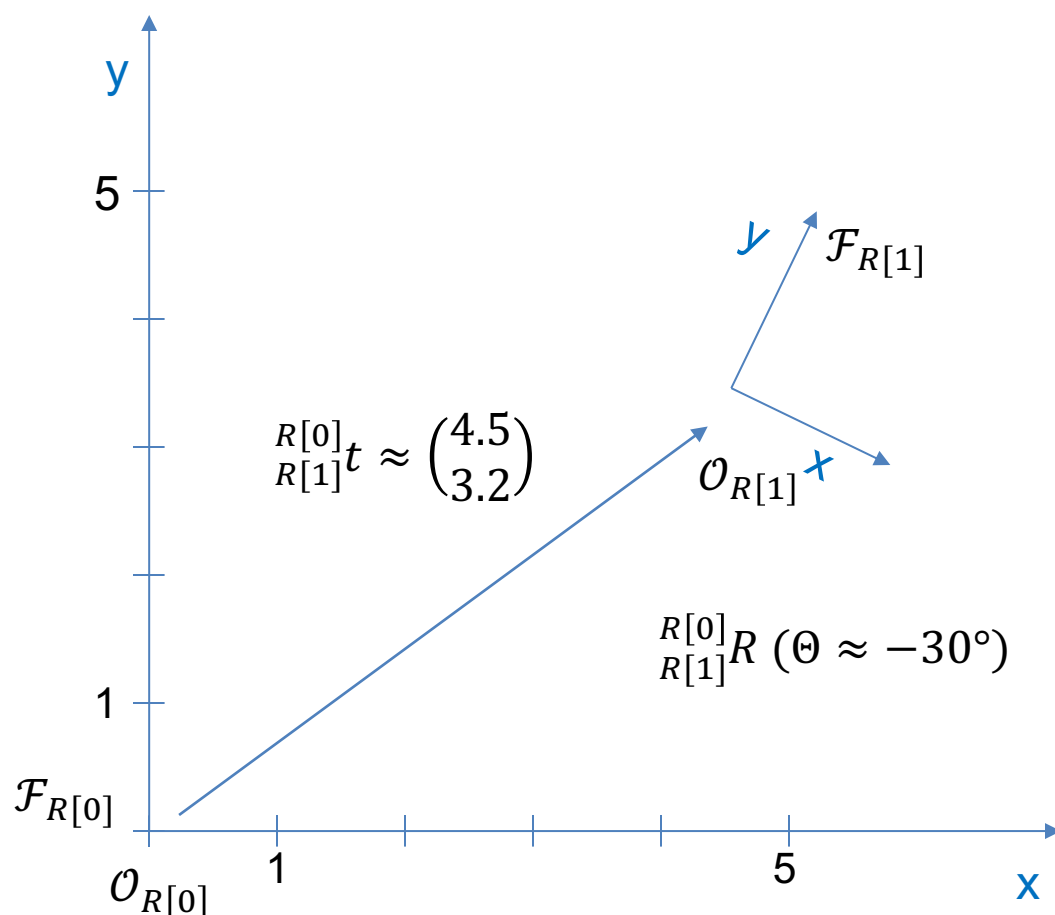
- **Position:**
 - $\begin{pmatrix} x \\ y \end{pmatrix}$ coordinates of any object or point (or another frame)
 - with respect to (wrt.) a specified frame
- **Orientation:**
 - (θ) angle of any oriented object (or another frame)
 - with respect to (wrt.) a specified frame
- **Pose:**
 - $\begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ position and orientation of any oriented object
 - with respect to (wrt.) a specified frame

Translation, Rotation & Transform



- **Translation:**
 - $\begin{pmatrix} x \\ y \end{pmatrix}$ difference, change, motion from one reference frame to another reference frame
- **Rotation:**
 - (Θ) difference in angle, rotation between one reference frame and another reference frame
- **Transform:**
 - $\begin{pmatrix} x \\ y \\ \Theta \end{pmatrix}$ difference, motion between one reference frame and another reference frame

Position & Translation, Orientation & Rotation



- $\mathcal{F}_{R[X]}$: Frame of reference of the robot at time X
- Where is that frame $\mathcal{F}_{R[X]}$?
 - Can only be expressed with respect to (wrt.) another frame (e.g. global Frame \mathcal{F}_G) =>
 - Pose of $\mathcal{F}_{R[X]}$ wrt. \mathcal{F}_G

- $O_{R[X]}$: Origin of $\mathcal{F}_{R[X]}$
 - $\overrightarrow{O_{R[X]}O_{R[X+1]}}$: **Position** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

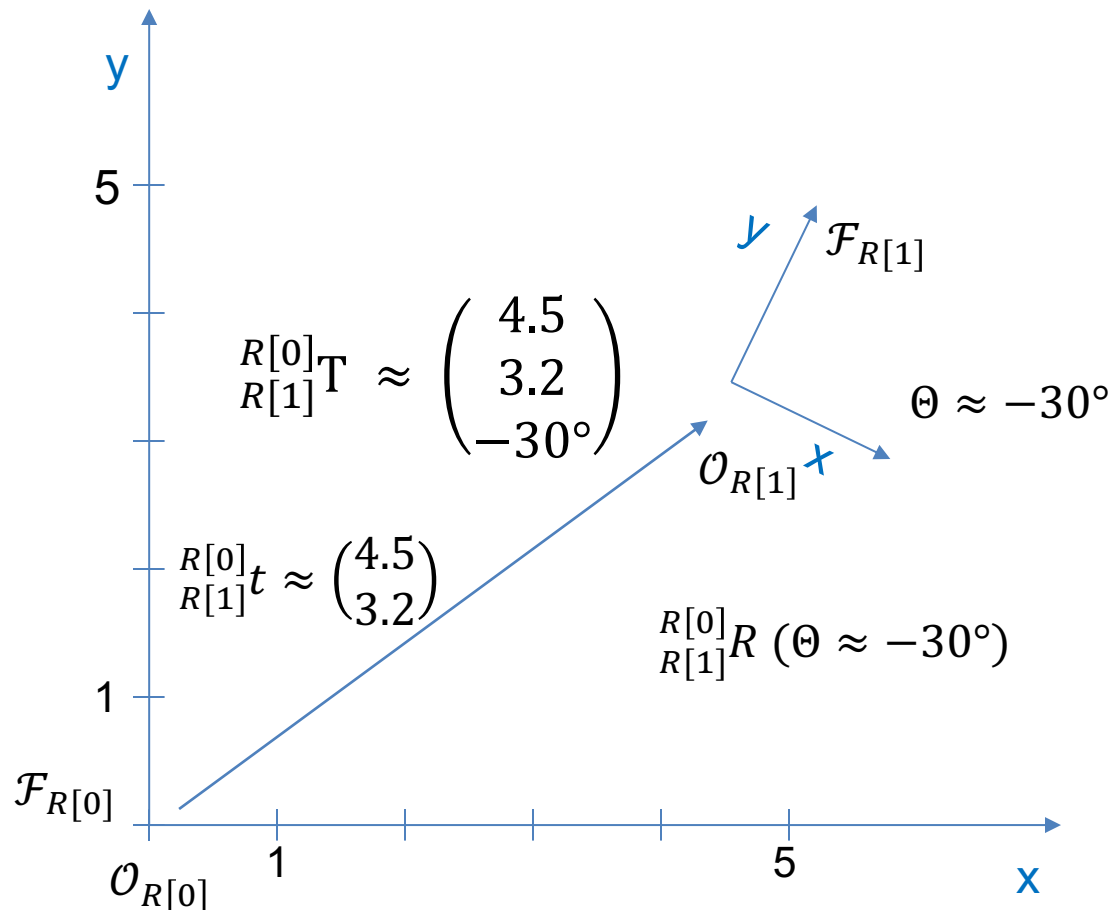
so $O_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

\triangleq ${}^{R[X]}_{R[X+1]}t$: **Translation**

- The angle θ between the x-Axes:
 - **Orientation** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

\triangleq ${}^{R[X]}_{R[X+1]}R$: **Rotation** of $\mathcal{F}_{R[X+1]}$ wrt. $\mathcal{F}_{R[X]}$

Transform



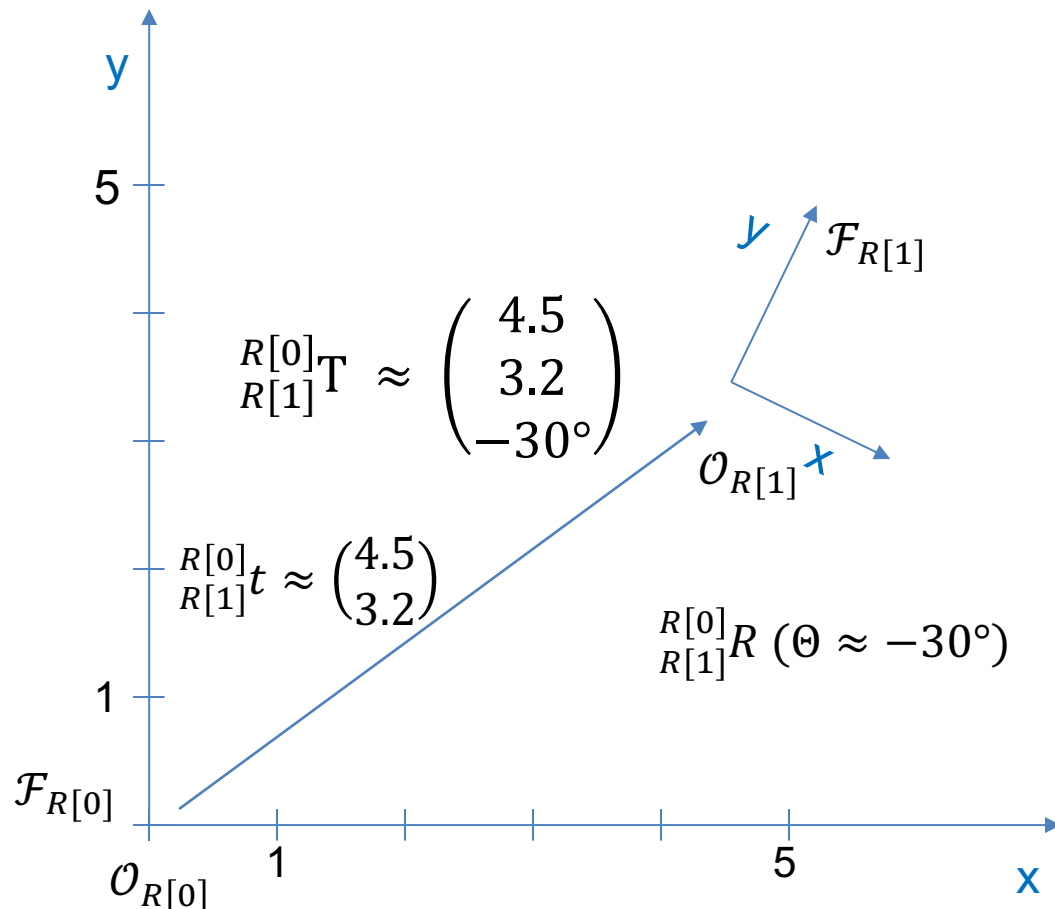
- $R_{R[X+1]}^{R[X]} t$: **Translation**
 - Position vector (x, y) of $R[X + 1]$ wrt. $R[X]$
- $R_{R[X+1]}^{R[X]} R$: **Rotation**
 - Angle (θ) of $R[X + 1]$ wrt. $R[X]$
- **Transform:** $R_{R[X+1]}^{R[X]} T \equiv \begin{Bmatrix} R_{R[X+1]}^{R[X]} t \\ R_{R[X+1]}^{R[X]} R \end{Bmatrix}$

Geometry approach to Odometry

We want to know:

- Position of the robot (x, y)
- Orientation of the robot (θ)
- => together: Pose $\begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$

With respect to (wrt.) \mathcal{F}_G : The global frame; global coordinate system

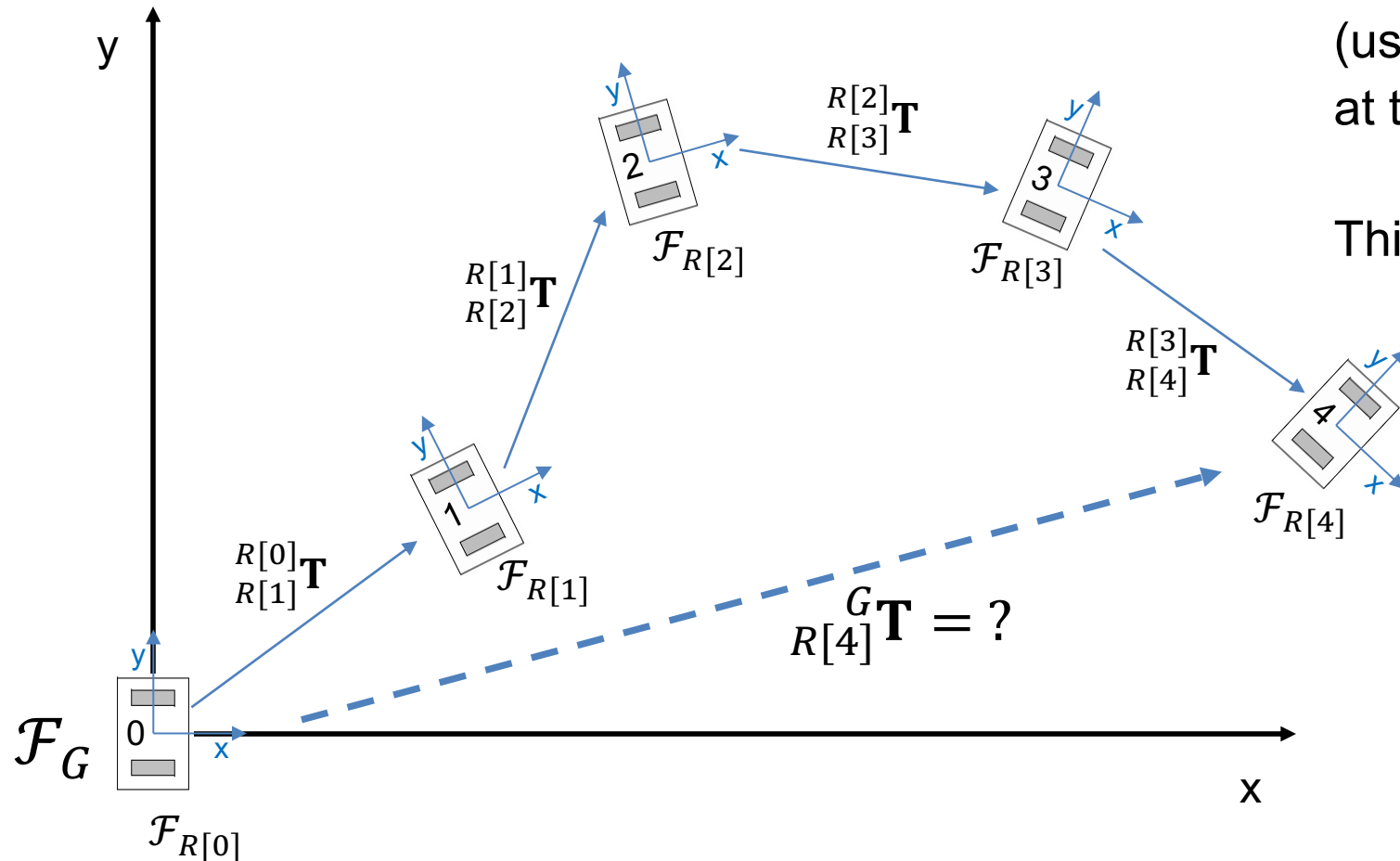


$$\mathcal{F}_{R[0]} = \mathcal{F}_G \Rightarrow {}^G\mathcal{F}_{R[0]} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$${}^G\mathcal{F}_{R[1]} = {}^{R[0]}T \approx \begin{pmatrix} 4.5 \\ 3.2 \\ 30^\circ \end{pmatrix}$$

Blackboard: ${}^{R[1]}T \approx \begin{pmatrix} 2 \\ 3 \\ 60^\circ \end{pmatrix}$

Mathematical approach: Transforms



Where is the Robot now?

The pose of $\mathcal{F}_{R[X]}$ with respect to \mathcal{F}_G (usually = $\mathcal{F}_{R[0]}$) is the pose of the robot at time X.

This is equivalent to ${}^G R_{R[X]} \mathbf{T}$

Chaining of Transforms

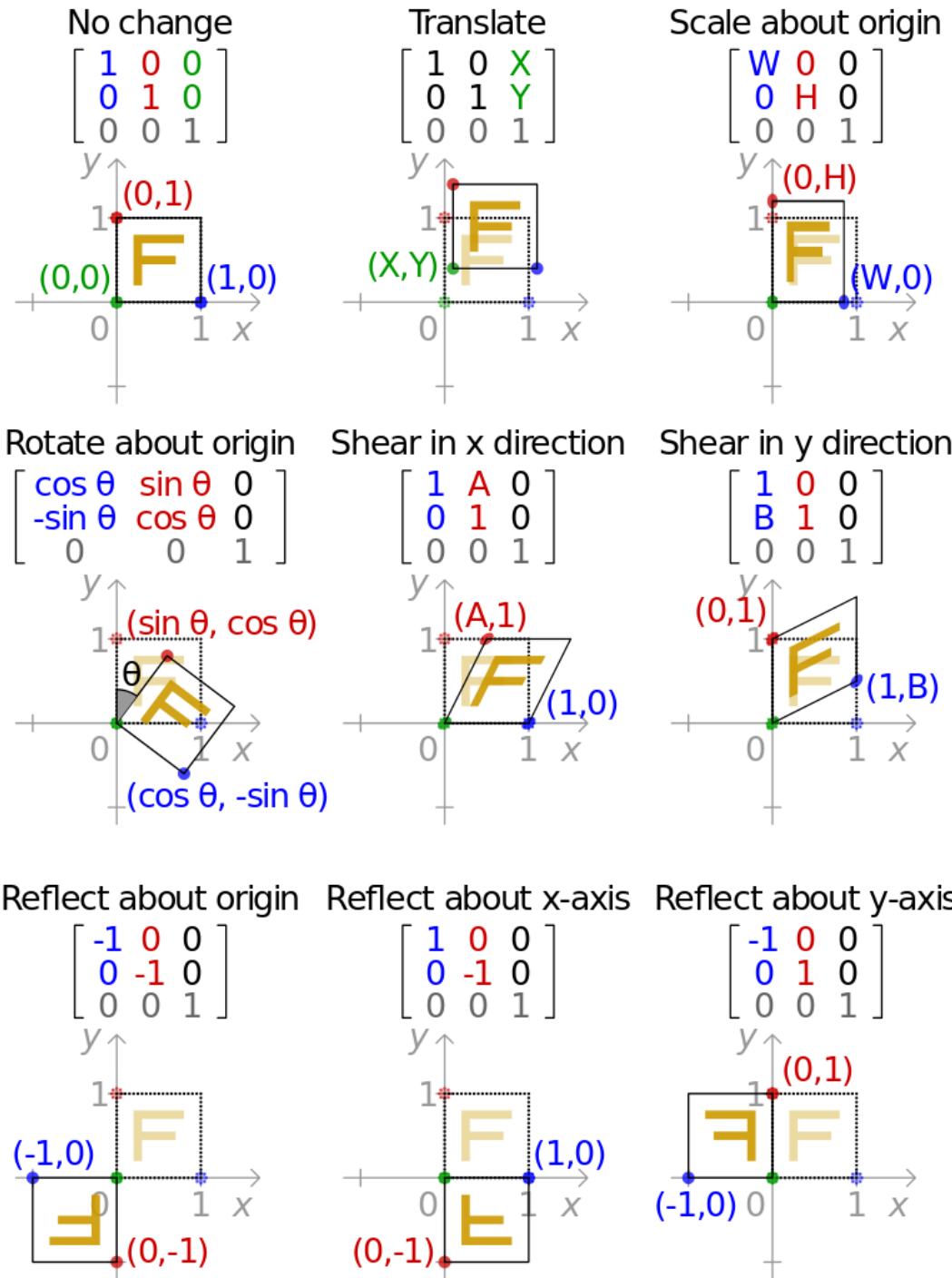
$${}^G R_{R[X+1]} \mathbf{T} = {}^G R_{R[X]} \mathbf{T} \quad {}^R R_{R[X+1]} \mathbf{T}$$

often: $\mathcal{F}_G \equiv \mathcal{F}_{R[0]} \Rightarrow {}^G R_{R[0]} \mathbf{T} = id$

Affine Transformation

- Function between affine spaces. Preserves:
 - points,
 - straight lines
 - planes
 - sets of parallel lines remain parallel
- Allows:
 - Interesting for Robotics: translation, rotation, (scaling), and chaining of those
 - Not so interesting for Robotics: reflection, shearing, homothetic transforms

- Rotation and Translation:
$$\begin{bmatrix} \cos \theta & \sin \theta & X \\ -\sin \theta & \cos \theta & Y \\ 0 & 0 & 1 \end{bmatrix}$$



Math: Rigid Transformation

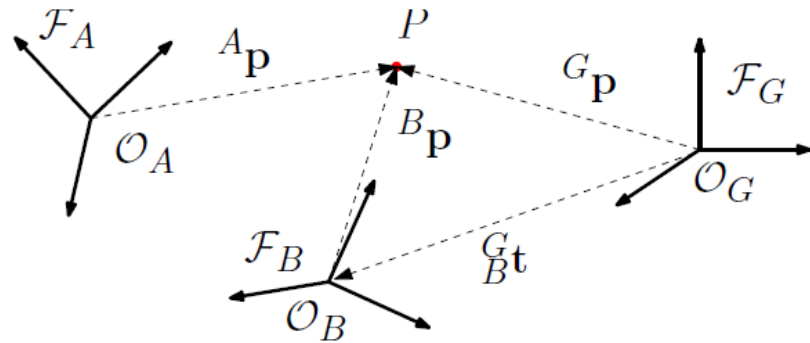
- Geometric transformation that preserves Euclidean distance between pairs of points.
- Includes reflections (i.e. change from right-hand to left-hand coordinate system and back)
- Just rotation & translation: rigid motions or proper rigid transformations:
 - Decomposed to rotation and translation
 - => subset of Affine Transformations
- In Robotics: Just use term **Transform** or **Transformation** for rigid motions

Lie groups for transformations

- Smoothly differentiable Group
- No singularities
- Good interpolation
- SO: Special Orthogonal group
- SE: Special Euclidian group
- Similarity transform group

| Group | Description | Dim. | Matrix Representation |
|--------|---|------|--|
| SO(3) | 3D Rotations | 3 | 3D rotation matrix |
| SE(3) | 3D Rigid transformations | 6 | Linear transformation on homogeneous 4-vectors |
| SO(2) | 2D Rotations | 1 | 2D rotation matrix |
| SE(2) | 2D Rigid transformations | 3 | Linear transformation on homogeneous 3-vectors |
| Sim(3) | 3D Similarity transformations (rigid motion + scale) | 7 | Linear transformation on homogeneous 4-vectors |

Transform



| Notation | Meaning |
|------------------------|--|
| $\mathcal{F}_{R[k]}$ | Coordinate frame attached to object 'R' (usually the robot) at sample time-instant k . |
| $O_{R[k]}$ | Origin of $\mathcal{F}_{R[k]}$. |
| ${}^{R[k]}p$ | For any general point P , the position vector $\overrightarrow{O_{R[k]}P}$ resolved in $\mathcal{F}_{R[k]}$. |
| ${}^H\hat{x}_R$ | The x-axis direction of \mathcal{F}_R resolved in \mathcal{F}_H . Similarly, ${}^H\hat{y}_R$, ${}^H\hat{z}_R$ can be defined. Obviously, ${}^R\hat{x}_R = \hat{e}_1$. Time indices can be added to the frames, if necessary. |
| ${}^{R[k]}S[{}^{k'}]R$ | The rotation-matrix of $\mathcal{F}_{S[{}^{k'}]}$ with respect to $\mathcal{F}_{R[k]}$. |
| ${}^R_S t$ | The translation vector $\overrightarrow{O_R O_S}$ resolved in \mathcal{F}_R . |

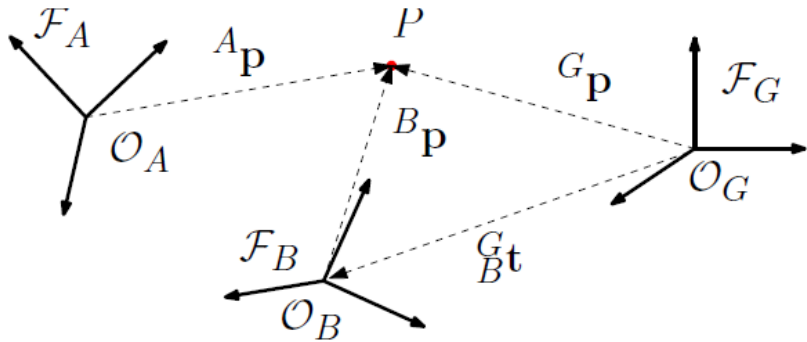
Transform
between two
coordinate frames

$${}^G_A t \triangleq \overrightarrow{O_G O_A} \text{ resolved in } \mathcal{F}_G \quad \begin{pmatrix} {}^G p \\ 1 \end{pmatrix} \equiv \begin{pmatrix} {}^G_A R & {}^G_A t \\ \mathbf{0}_{1 \times [2,3]} & 1 \end{pmatrix} \begin{pmatrix} {}^A p \\ 1 \end{pmatrix} \quad {}^G_A T \equiv \left\{ \begin{matrix} {}^G_A t \\ {}^G_A R \end{matrix} \right\}$$

$${}^G p = {}^G_A R \cdot {}^A p + {}^G_A t \\ \triangleq {}^G_A T ({}^A p).$$

$$\begin{bmatrix} \cos \theta & -\sin \theta & {}^G_A t_x \\ \sin \theta & \cos \theta & {}^G_A t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Transform: Operations



Transform between two coordinate frames (chaining, compounding):

$${}^G\mathbf{T} = {}^G\mathbf{T} {}^A\mathbf{T} \equiv \begin{Bmatrix} {}^G\mathbf{R} {}^A\mathbf{t} + {}^G\mathbf{t} \\ {}^G\mathbf{R} {}^A\mathbf{R} \end{Bmatrix}$$

Inverse of a Transform :

$${}^B\mathbf{T} = {}^A\mathbf{T}^{-1} \equiv \begin{Bmatrix} -{}^A\mathbf{R}^T {}^A\mathbf{t} \\ {}^A\mathbf{R}^T \end{Bmatrix}$$

Relative (Difference) Transform : ${}^B\mathbf{T} = {}^G\mathbf{T}^{-1} {}^G\mathbf{T}$

See: **Quick Reference to Geometric Transforms in Robotics** by Kaustubh Pathak on the webpage!

Chaining :
$${}_{R[X+1]}{}^G\mathbf{T} = {}_{R[X]}{}^G\mathbf{T} \quad {}_{R[X+1]}{}^{R[X]}\mathbf{T} \equiv \begin{Bmatrix} {}_{R[X]}{}^G\mathbf{R} & {}_{R[X+1]}{}^{R[X]}t + {}_{R[X]}{}^Gt \\ {}_{R[X]}{}^G\mathbf{R} & {}_{R[X+1]}{}^{R[X]}\mathbf{R} \end{Bmatrix} = \begin{Bmatrix} {}_{R[X+1]}{}^Gt \\ {}_{R[X+1]}{}^G\mathbf{R} \end{Bmatrix}$$

In 2D Translation:
$$\begin{bmatrix} {}_{R[X+1]}{}^Gt_x \\ {}_{R[X+1]}{}^Gt_y \\ 1 \end{bmatrix} = \begin{bmatrix} \cos {}_{R[X]}{}^G\theta & -\sin {}_{R[X]}{}^G\theta & {}_{R[X]}{}^Gt_x \\ \sin {}_{R[X]}{}^G\theta & \cos {}_{R[X]}{}^G\theta & {}_{R[X]}{}^Gt_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}_{R[X]}{}^{R[X]}t_x \\ {}_{R[X]}{}^{R[X]}t_y \\ 1 \end{bmatrix}$$

In 2D Rotation:

$${}_{R[X+1]}{}^G\mathbf{R} = \begin{bmatrix} \cos {}_{R[X+1]}{}^G\theta & -\sin {}_{R[X+1]}{}^G\theta \\ \sin {}_{R[X+1]}{}^G\theta & \cos {}_{R[X+1]}{}^G\theta \end{bmatrix} = \begin{bmatrix} \cos {}_{R[X]}{}^G\theta & -\sin {}_{R[X]}{}^G\theta \\ \sin {}_{R[X]}{}^G\theta & \cos {}_{R[X]}{}^G\theta \end{bmatrix} \begin{bmatrix} \cos {}_{R[X+1]}{}^{R[X]}\theta & -\sin {}_{R[X+1]}{}^{R[X]}\theta \\ \sin {}_{R[X+1]}{}^{R[X]}\theta & \cos {}_{R[X+1]}{}^{R[X]}\theta \end{bmatrix}$$

In 2D Rotation (simple):
$${}_{R[X+1]}{}^G\theta = {}_{R[X]}{}^G\theta + {}_{R[X+1]}{}^{R[X]}\theta$$

In ROS: nav_2d_msgs/Pose2DStamped

- First Message at time 97 : G
- Message at time 103 : X
- Next Message at time 107 : X+1

$$R_{[X+1]}^G \mathbf{T} = R_{[X]}^G \mathbf{T} R_{[X+1]}^{R[X]} \mathbf{T}$$

$$R_{[X]} \mathbf{T} \begin{matrix} t_x \\ t_y \end{matrix}$$

$$R_{[X+1]} \mathbf{T} \begin{matrix} t_x \\ t_y \end{matrix}$$

$$R_{[X]} \Theta$$

$$R_{[X+1]} \Theta$$

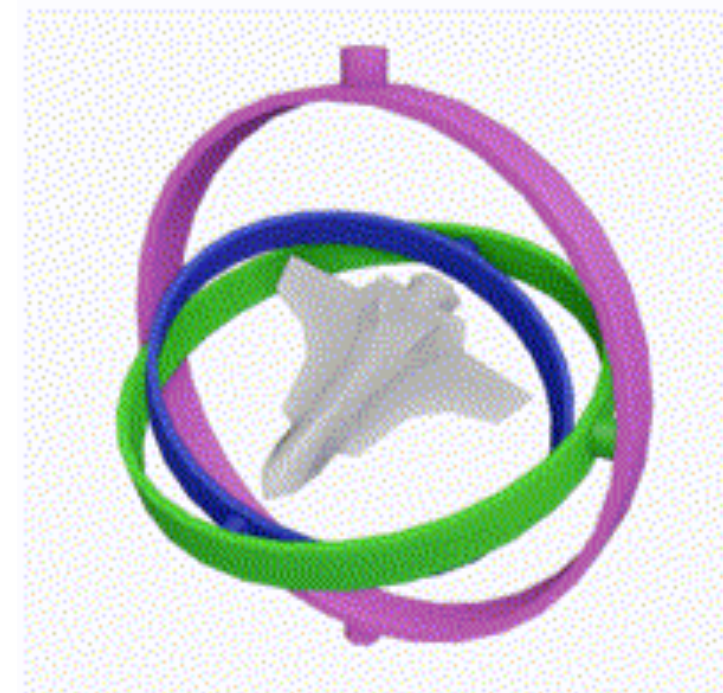
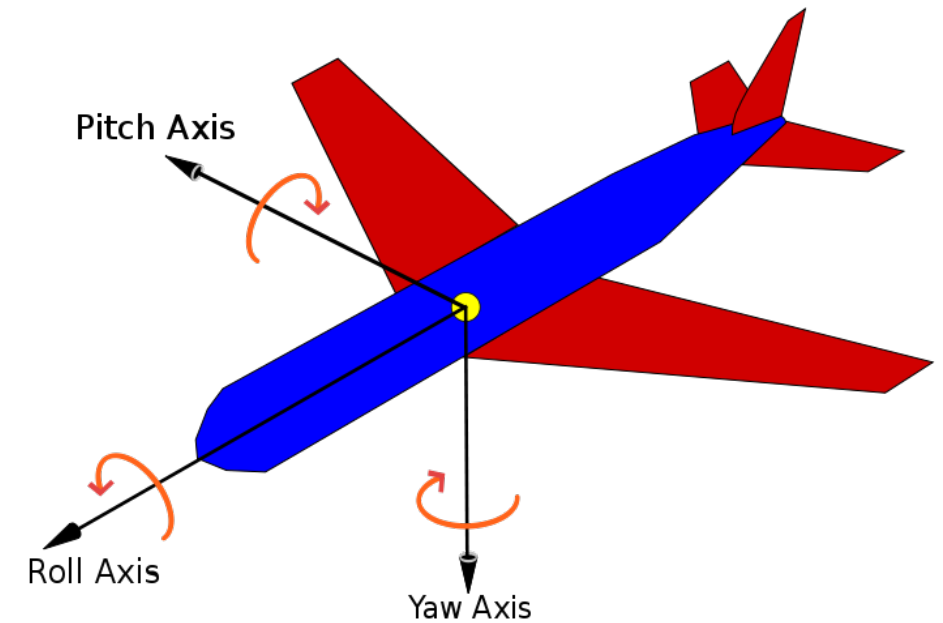
```
std_msgs/Header header
  uint32 seq
  time stamp
  string frame_id
geometry_msgs/Pose2D pose2D
  float64 x
  float64 y
  float64 theta
```

3D Rotation

- Many 3D rotation representations:

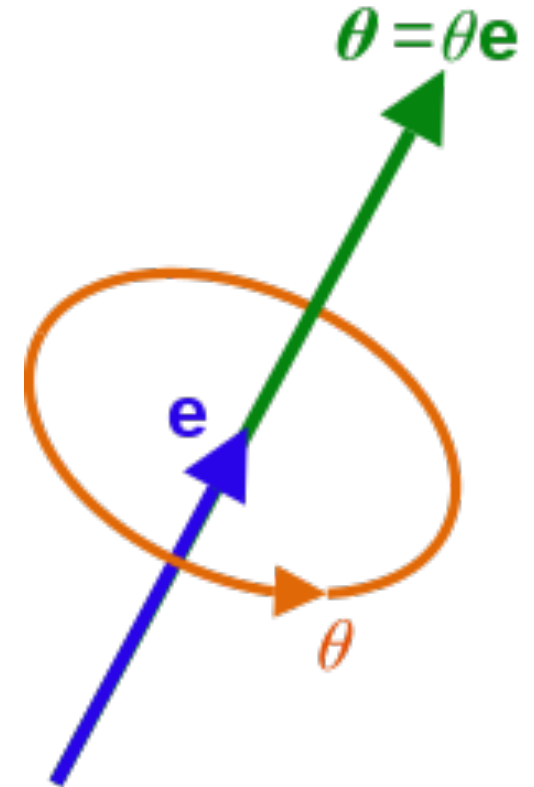
https://en.wikipedia.org/wiki/Rotation_formalisms_in_three_dimensions

- Euler angles:
 - Roll: rotation around x-axis
 - Pitch: rotation around y-axis
 - Yaw: rotation around z-axis
 - Apply rotations one after the other...
 - => Order important! E.g.:
 - x-z-x; x-y-z; z-y-x; ...
 - ☹ Singularities
 - Gimbal lock in Engineering
 - "a condition caused by the collinear alignment of two or more robot axes resulting in unpredictable robot motion and velocities"



3D Rotation

- Axis Angle
 - Angle θ and
 - Axis unit vector \mathbf{e} (3D vector with length 1)
 - Can be represented with 2 numbers (e.g. elevation and azimuth angles)
- Euler Angles: sequence of 3 rotations around coordinate axes
equivalent to:
- Axis Angle: pure rotation around a single fixed axis



3D Rotation

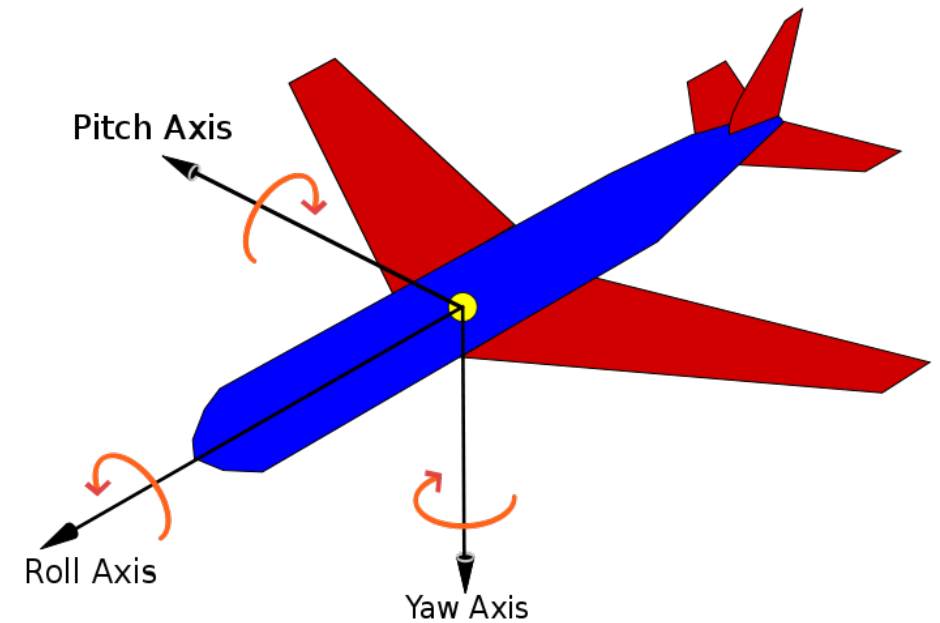
- Quaternions:
 - Concatenating rotations is computationally faster and numerically more stable
 - Extracting the angle and axis of rotation is simpler
 - Interpolation is more straightforward
 - Unit Quaternion: norm = 1

- Versor: <https://en.wikipedia.org/wiki/Versor>

https://en.wikipedia.org/wiki/Quaternions_and_spatial_rotation

- Scalar (real) part: q_0 , sometimes q_w
- Vector (imaginary) part: \mathbf{q}
- Over determined: 4 variables for 3 DoF (but: unit!)

- Check out: <https://eater.net/quaternions> !



$$\check{\mathbf{p}} \equiv p_0 + p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}$$

$$\check{\mathbf{q}} = (q_0 \quad q_x \quad q_y \quad q_z)^T \equiv \begin{pmatrix} q_0 \\ \mathbf{q} \end{pmatrix}$$

Transform in 3D

$${}^G\mathbf{T}_A = \begin{matrix} & \text{Matrix} & \text{Euler} & \text{Quaternion} \\ = & \begin{bmatrix} {}^G\mathbf{R}_A & {}^G\mathbf{t}_A \\ \mathbf{0}_{1 \times 3} & 1 \end{bmatrix} & \begin{pmatrix} {}^G\mathbf{t}_A \\ {}^G\Theta_A \end{pmatrix} & \begin{pmatrix} {}^G\mathbf{t}_A \\ {}^G\check{\mathbf{q}}_A \end{pmatrix} \end{matrix}$$

$${}^G\Theta_A \triangleq (\theta_r, \theta_p, \theta_y)^T$$

In ROS: Quaternions! (w, x, y, z)
Uses Eigen library for Transforms

Rotation Matrix 3x3

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_z(\alpha) R_y(\beta) R_x(\gamma)$$

yaw = α , pitch = β , roll = γ

Eigen

- Don't have to deal with the details of transforms too much 😊
- Conversions between ROS and Eigen:
http://docs.ros.org/noetic/api/eigen_conversions/html/namespace_tf.html

```
Matrix3f m;  
m = AngleAxisf(angle1, Vector3f::UnitZ())  
    * AngleAxisf(angle2, Vector3f::UnitY())  
    * AngleAxisf(angle3, Vector3f::UnitZ());
```

https://eigen.tuxfamily.org/dox/group_Geometry_Module.html

| | | |
|-------|---|---|
| class | Eigen::AlignedBox | An axis aligned box. More... |
| class | Eigen::AngleAxis | Represents a 3D rotation as a rotation angle around an arbitrary 3D axis. More... |
| class | Eigen::Homogeneous | Expression of one (or a set of) homogeneous vector(s) More... |
| class | Eigen::Hyperplane | A hyperplane. More... |
| class | Eigen::Map< const Quaternion< _Scalar >, _Options > Quaternion | expression mapping a constant memory buffer. More... |
| class | Eigen::Map< Quaternion< _Scalar >, _Options > | Expression of a quaternion from a memory buffer. More... |
| class | Eigen::ParametrizedLine | A parametrized line. More... |
| class | Eigen::Quaternion | The quaternion class used to represent 3D orientations and rotations. More... |
| class | Eigen::QuaternionBase | Base class for quaternion expressions. More... |
| class | Eigen::Rotation2D | Represents a rotation/orientation in a 2 dimensional space. More... |
| class | Scaling | Represents a generic uniform scaling transformation. More... |
| class | Eigen::Transform | Represents an homogeneous transformation in a N dimensional space. More... |
| class | Eigen::Translation | Represents a translation transformation. More... |

Examples of Transforms

- Transform between global coordinate frame and robot frame at time X
- Transform between robot frame at time X and robot frame at time $X+1$
- Transform between robot camera frame and robot base frame (mounted fixed – not dependend on time! => static transform)
- Transform between map origin and door pose in map (not time dependend)
- Transform between robot camera frame and fingers (end-effector) of a robot arm at time X
- Transform between robot camera frame and map frame at time X
- Transform between robot 1 camera at time X and robot 2 camera at time $X+n$

ROS Standards:

- Standard Units of Measure and Coordinate Conventions
 - <http://www.ros.org/reps/rep-0103.html>
- Coordinate Frames for Mobile Platforms:
 - <http://www.ros.org/reps/rep-0105.html>