上海科技大学
ShanghaiTech University

CS283: **Robotics Spring 2023: SLAM II**

Sören Schwertfeger

ShanghaiTech University

LAST LECTURE

Bayes Filter

State Estimation

- Estimate the state x of a system given observations z and controls u

- **Goal:**

$$p(x_t | z_{1:t}, u_{1:t})$$

Recursive Bayes Filter 1

$$bel(x_t) = \underline{p(x_t \mid z_{1:t}, u_{1:t})}$$

definition of the belief

Recursive Bayes Filter 2

$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \underline{\eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}
\end{aligned}
$$

Bayes’ rule

Complete Derivation of the Recursive Bayes Filter

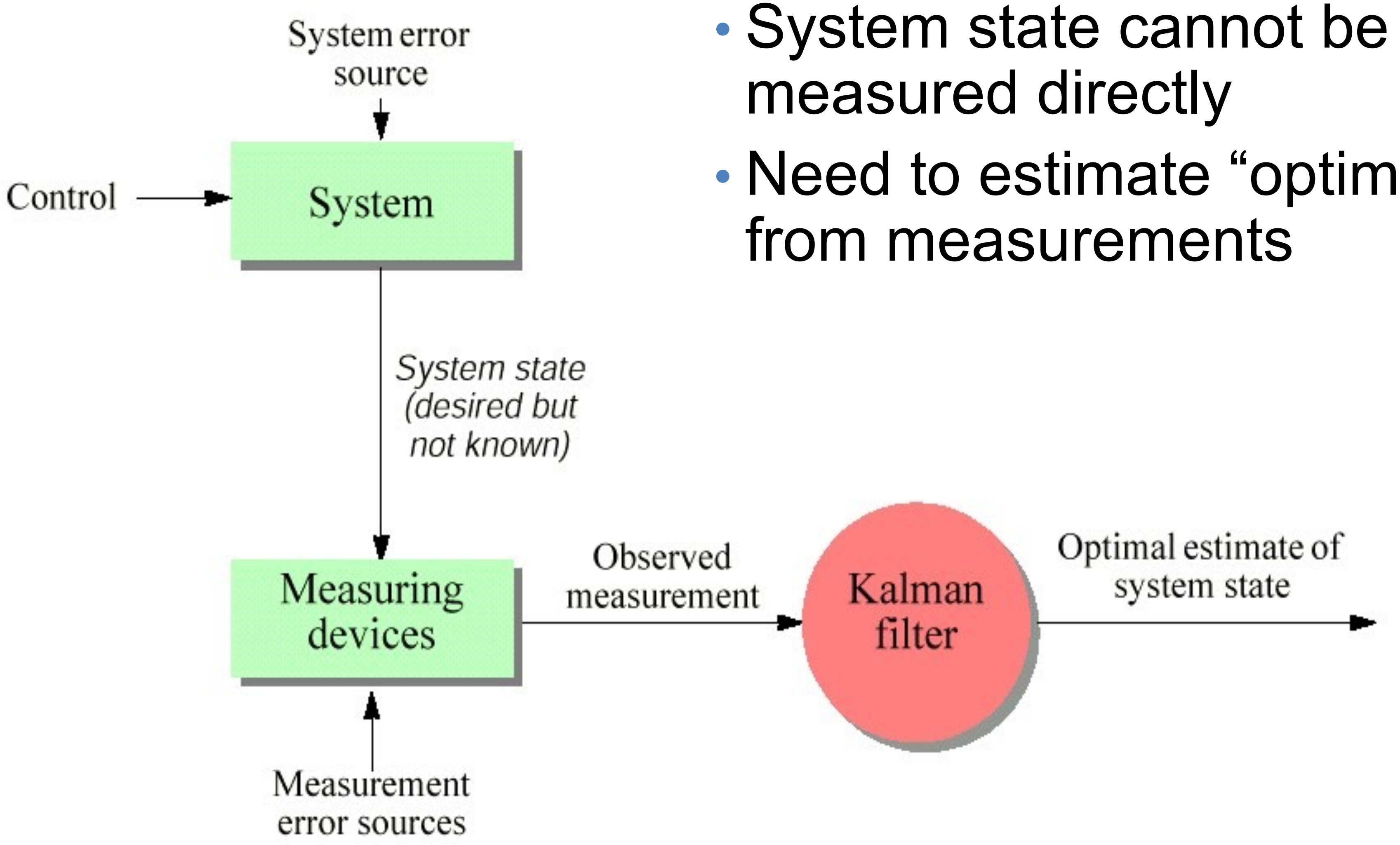
$$
\begin{aligned}
bel(x_t) &= p(x_t \mid z_{1:t}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t)\, p(x_t \mid z_{1:t-1}, u_{1:t}) \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) \\
&\qquad\qquad p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1} \\
&= \eta\, p(z_t \mid x_t) \int p(x_t \mid x_{t-1}, u_t)\, bel(x_{t-1})\, dx_{t-1}
\end{aligned}
$$

KALMAN FILTER DETAILS

Following Material:

- Michael Williams, Australian National University
- Cornelia Fermüller, University of Maryland

The Problem



- System state cannot be measured directly
- Need to estimate “optimally” from measurements

Theoretical Basis

Prediction (Time Update)

(1) Project the state ahead

$$\hat{y}^-_k = Ay_{k-1} + Bu_k$$

(2) Project the error covariance ahead

$$P^-_k = AP_{k-1}A^T + Q$$

Correction (Measurement Update)

(1) Compute the Kalman Gain

$$K = P^-_kH^T(HP^-_kH^T + R)^{-1}$$

(2) Update estimate with measurement z_k

$$\hat{y}_k = \hat{y}^-_k + K(z_k - H\,\hat{y}^-_k)$$

(3) Update Error Covariance

$$P_k = (I - KH)P^-_k$$

Bayes Filter Reminder

- Prediction

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

- Correction

$$bel(x_t) = \eta\, p(z_t \mid x_t)\, \overline{bel}(x_t)$$

Kalman Filter

- Bayes filter with **Gaussians**
- Developed in the late 1950's
- Most relevant Bayes filter variant in practice
- Applications range from economics, wheather forecasting, satellite navigation to robotics and many more.

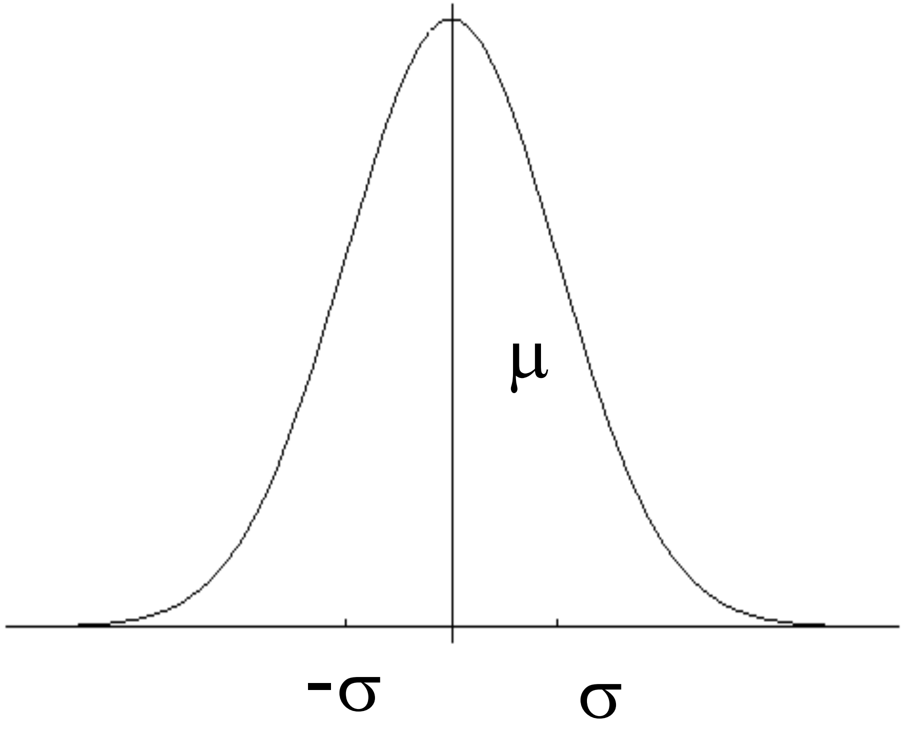
- The Kalman filter "algorithm" is a couple of **matrix multiplications**!

Gaussians

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

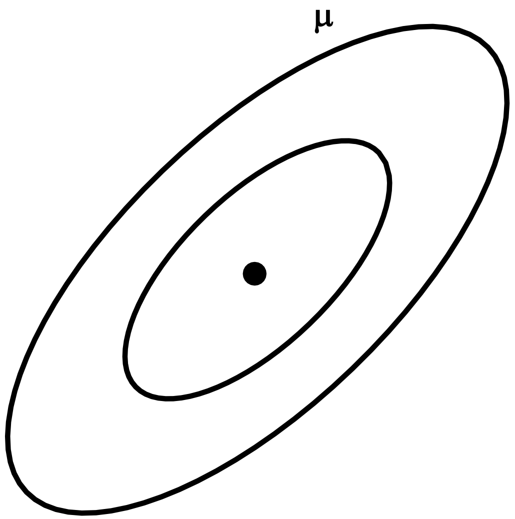
Univariate



$$p(\mathbf{x}) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}):$$

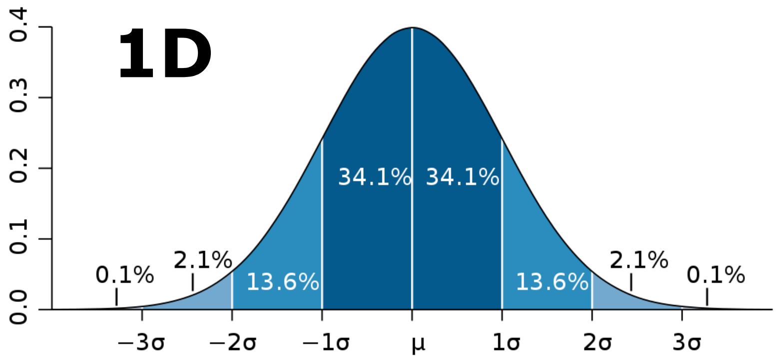
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^t \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

Multivariate

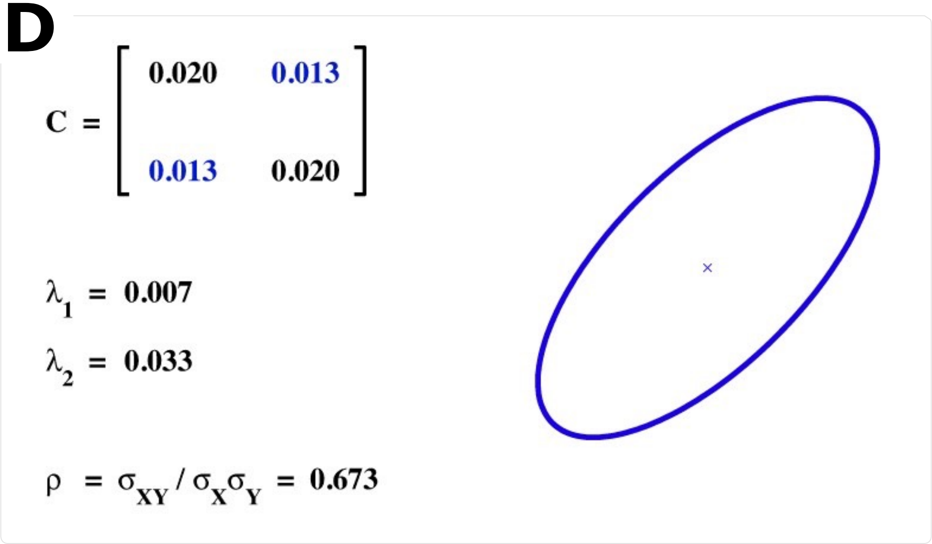


Gaussians

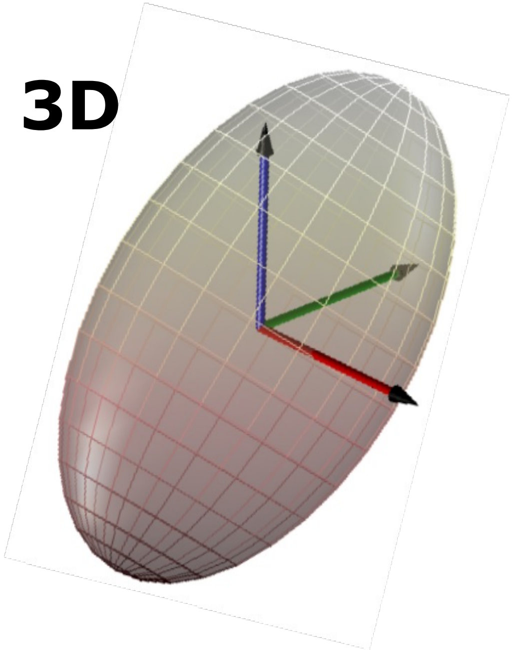
1D



2D



3D



Properties of Gaussians

- Univariate

$$\left.\begin{array}{l} X \sim N(\mu,\sigma^2) \\ Y = aX + b \end{array}\right\} \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1,\sigma_1^{\ 2}) \\ X_2 \sim N(\mu_2,\sigma_2^{\ 2}) \end{array}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\sigma_2^{\ 2}}{\sigma_1^{\ 2}+\sigma_2^{\ 2}}\mu_1 + \frac{\sigma_1^{\ 2}}{\sigma_1^{\ 2}+\sigma_2^{\ 2}}\mu_2, \quad \frac{1}{\sigma_1^{\ -2}+\sigma_2^{\ -2}}\right)$$

- Multivariate

$$\left.\begin{array}{l} X \sim N(\mu,\Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

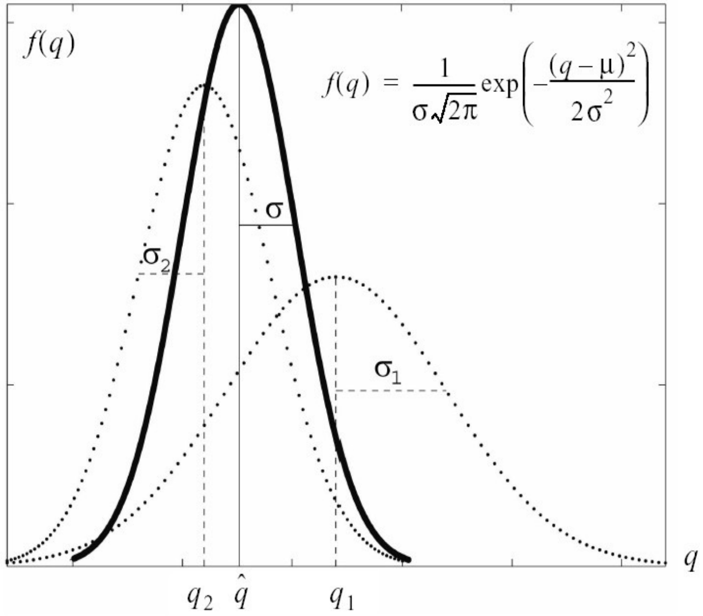
$$\left.\begin{array}{l} X_1 \sim N(\mu_1,\Sigma_1) \\ X_2 \sim N(\mu_2,\Sigma_2) \end{array}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1+\Sigma_2}\mu_1 + \frac{\Sigma_1}{\Sigma_1+\Sigma_2}\mu_2, \quad \frac{1}{\Sigma_1^{-1}+\Sigma_2^{-1}}\right)$$

- We stay in the "Gaussian world" as long as we start with Gaussians and perform only linear transformations

Introduction to Kalman Filter (1)

- Two measurements no dynamics

$$\hat{q}_1 = q_1 \text{ with variance } \sigma_1^2$$
$$\hat{q}_2 = q_2 \text{ with variance } \sigma_2^2$$



- Weighted least-square

$$S = \sum_{i=1}^{n} w_i(\hat{q} - q_i)^2$$

- Finding minimum error

$$\frac{\partial S}{\partial \hat{q}} = \frac{\partial}{\partial \hat{q}}\sum_{i=1}^{n} w_i(\hat{q} - q_i)^2 = 2\sum_{i=1}^{n} w_i(\hat{q} - q_i) = 0$$

- After some calculation and rearrangements

$$\hat{q} = q_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(q_2 - q_1)$$

- Another way to look at it – weighted mean

Discrete Kalman Filter

- Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

Process dynamics

- with a measurement

$$z_t = C_t x_t + \delta_t$$

Observation model

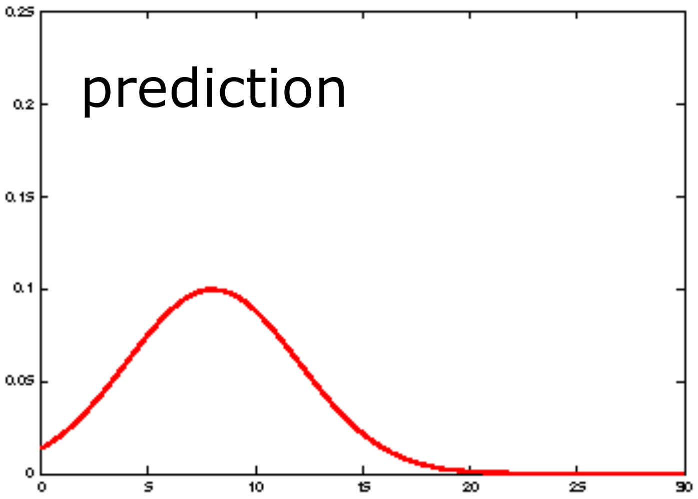
A_t Matrix (nxn) that describes how the state evolves from t - 1 to t without controls or noise.

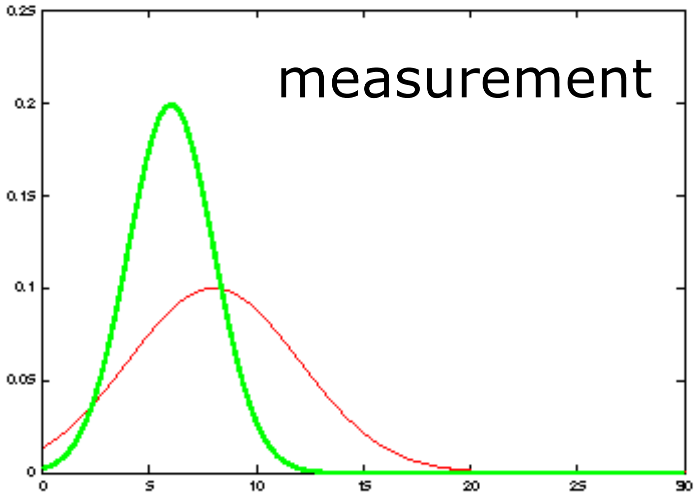
B_t Matrix (nxl) that describes how the control u_t changes the state from t-1 to t.

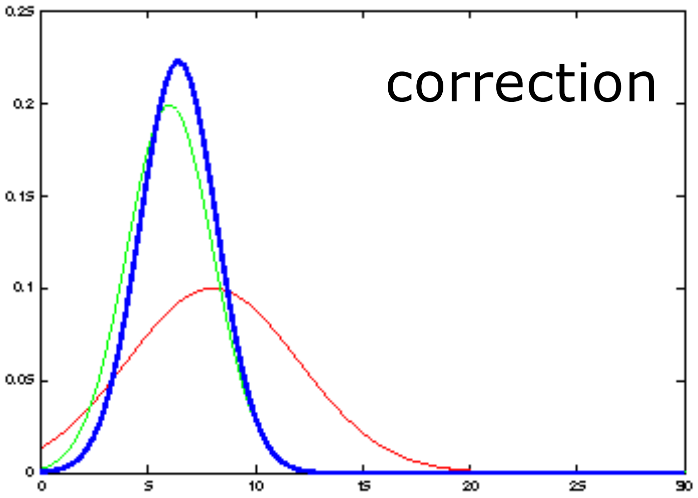
C_t Matrix (kxn) that describes how to map the state x_t to an observation z_t.

ε_t, δ_t Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.

Kalman Filter Updates in 1D







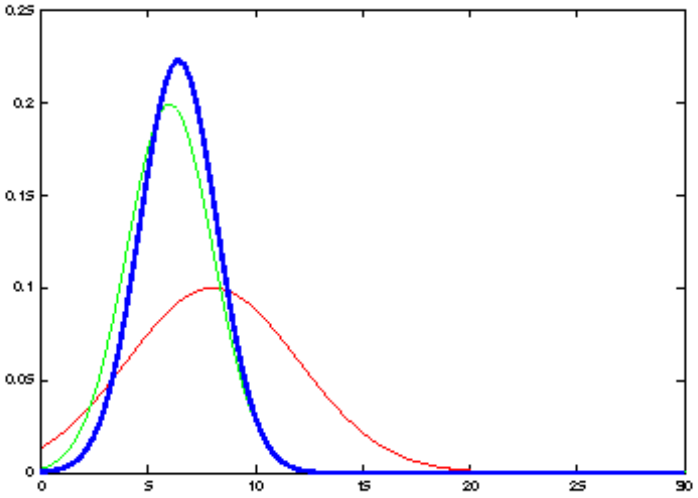
It's a weighted mean!

Kalman Filter Updates in 1D

gain innovation

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases} \quad \text{with} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

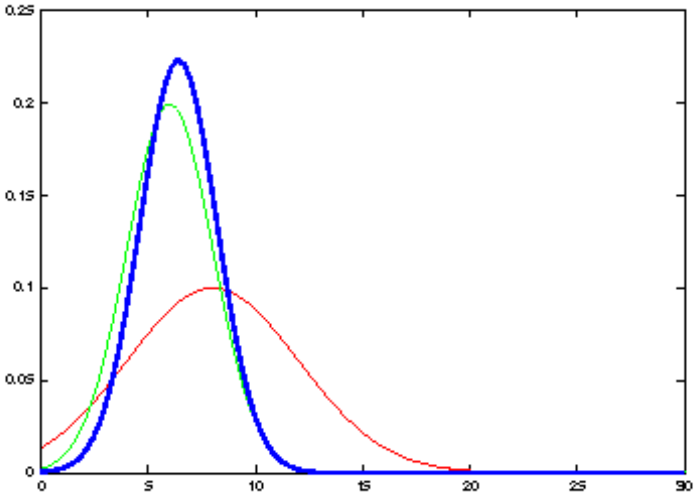
$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_tC_t)\bar{\Sigma}_t \end{cases} \quad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

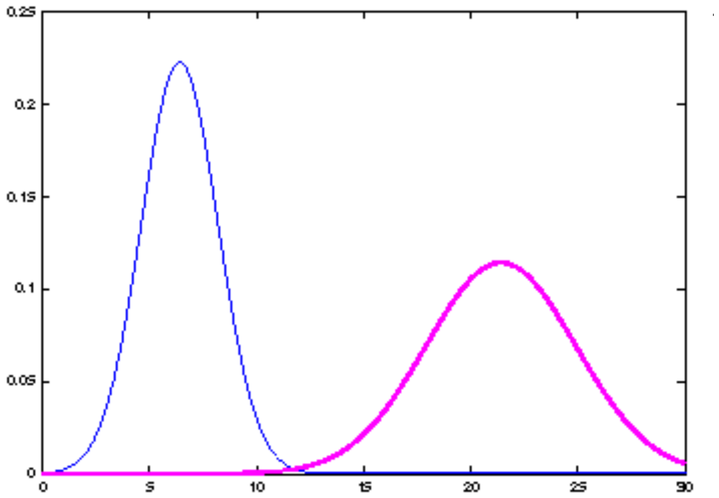


Kalman Filter Updates in 1D

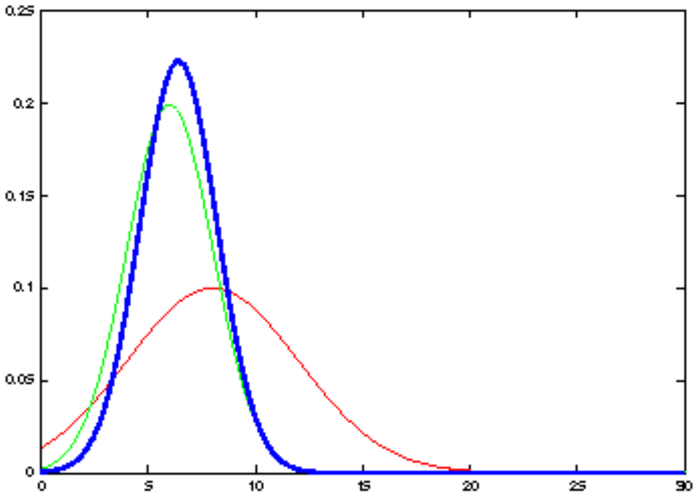
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

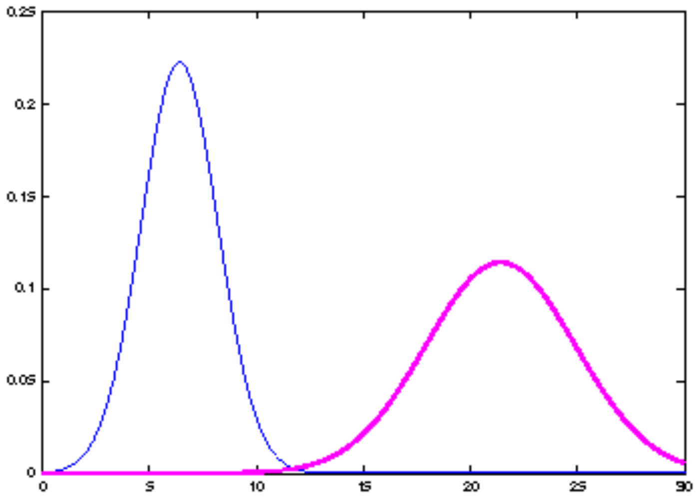
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

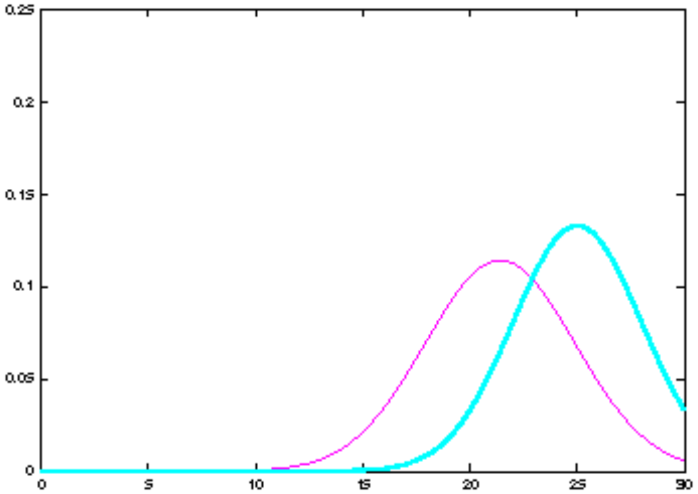


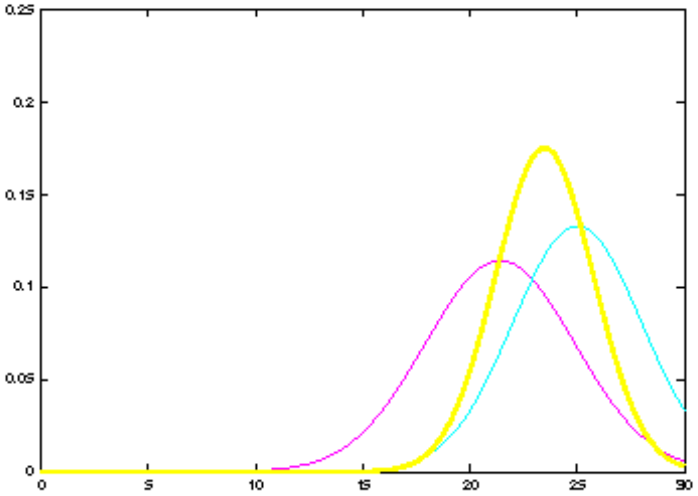


Kalman Filter Updates









Kalman Filter Algorithm

1. Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

   Prediction:

2. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
3. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

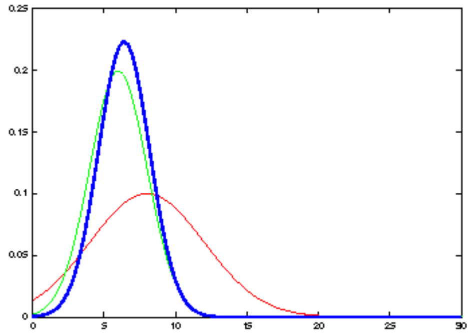
   Correction:

4. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
5. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
6. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

7. Return μ_t, Σ_t

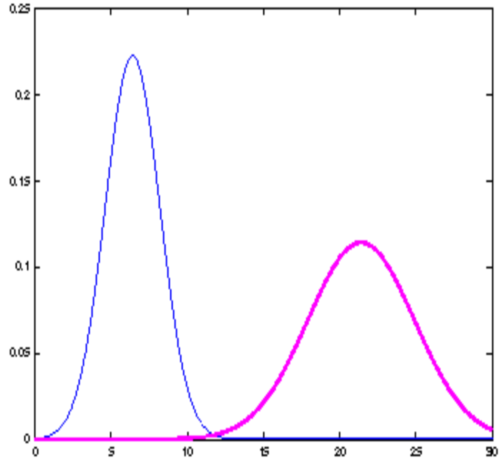
The Prediction-Correction-Cycle

Prediction

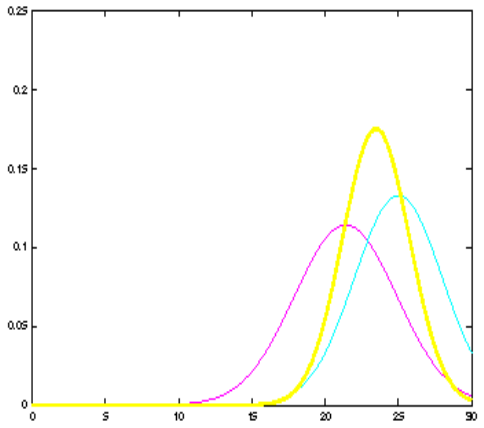


$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

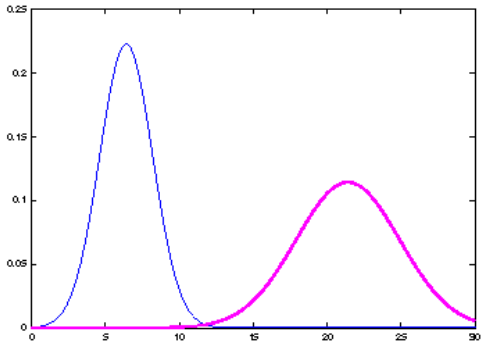


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \mu_t \quad K_t(z_t \quad \mu_t) \\ \sigma_t^2 = (1 \quad K_t)\sigma_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{1}$$



Correction

The Prediction-Correction-Cycle

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

Kalman Filter Summary

- Highly efficient: Polynomial in measurement dimensionality k and state dimensionality n:

$$O(k^{2.376} + n^2)$$

- Optimal for linear Gaussian systems!
- Most robotics systems are nonlinear!

EXTENDED KALMAN FILTER (EKF)

Following Material:

- Cyrill Stachniss, University of Bonn

Non-linear Dynamic Systems

- Most realistic problems (in robotics) involve nonlinear functions

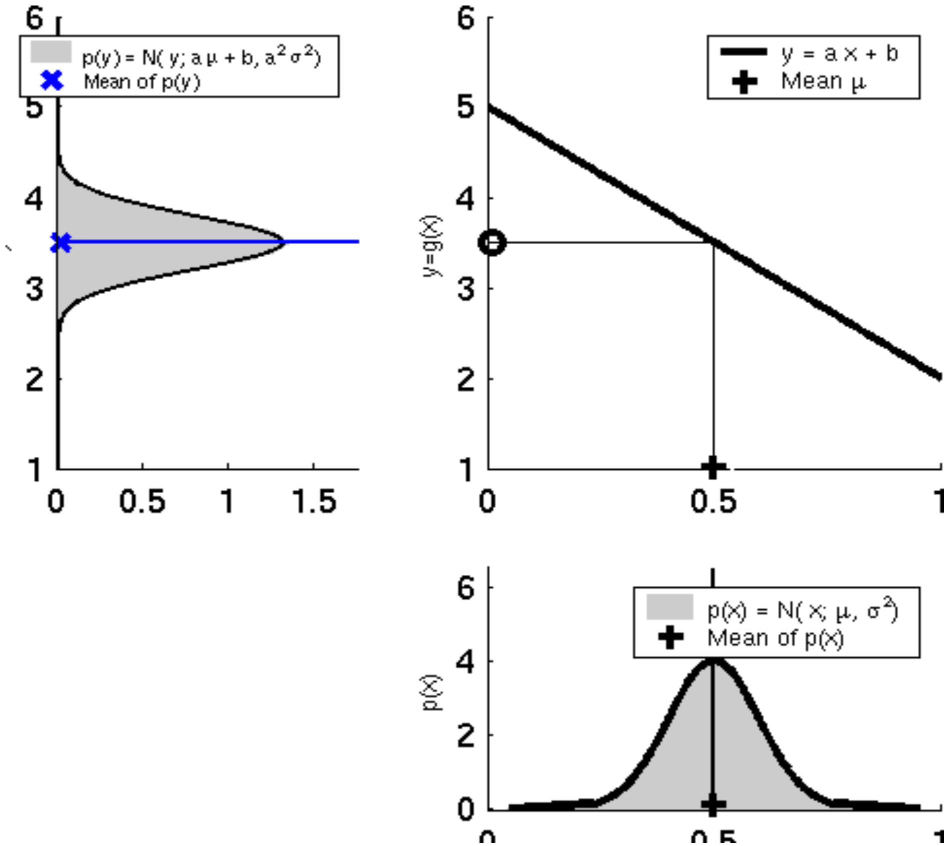
$$\cancel{x_t = A_t x_{t-1} + B_t u_t + \epsilon_t} \qquad \cancel{z_t = C_t x_t + \delta_t}$$

$$\Downarrow \qquad\qquad \Downarrow$$

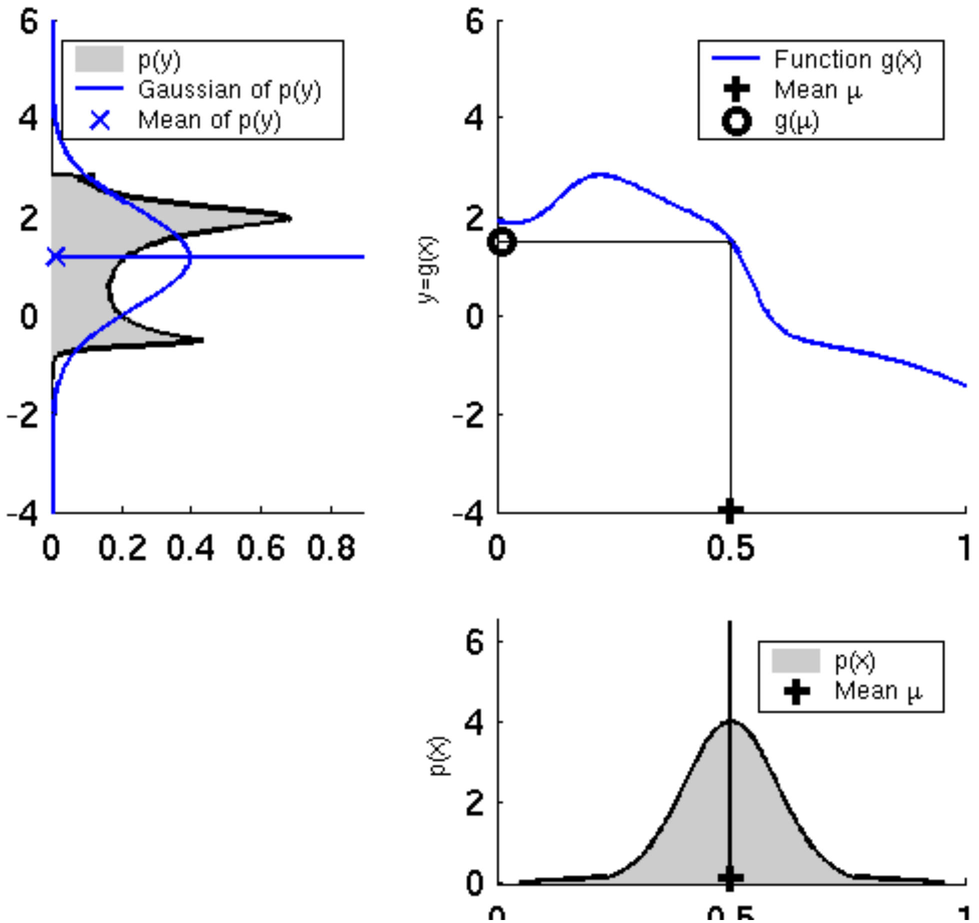
$$x_t = g(u_t, x_{t-1}) + \epsilon_t \qquad z_t = h(x_t) + \delta_t$$

- Extended Kalman filter relaxes linearity assumption

Linearity Assumption Revisited



Courtesy: Thrun, Burgard, Fox


p(y)
Gaussian of p(y)
Mean of p(y)
6
4
2
0
-2
-4
0 0.2 0.4 0.6 0.8
Function g(x)
Mean μ
g(μ)
y=g(x)
0 0.5 1
p(x)
Mean μ
p(x)

Other Error Prop. Techniques

- **Second-Order Error Propagation**

Rarely used (complex expressions)

- **Monte-Carlo**

Non-parametric representation of uncertainties

1. Sampling from $p(X)$
2. Propagation of samples
3. Histogramming
4. Normalization

- **Extended Kalman Filter: EKF**

EKF Linearization: First Order Taylor Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \underbrace{\frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}}_{=:\, G_t} (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \underbrace{\frac{\partial h(\bar{\mu}_t)}{\partial x_t}}_{=:\, H_t} (x_t - \bar{\mu}_t)$$

Jacobian matrices

Jacobian Matrix

- It's a **non-square matrix** $n \times m$ in general
- Suppose you have a vector-valued function $f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$
- Let the **gradient operator** be the vector of (first-order) partial derivatives

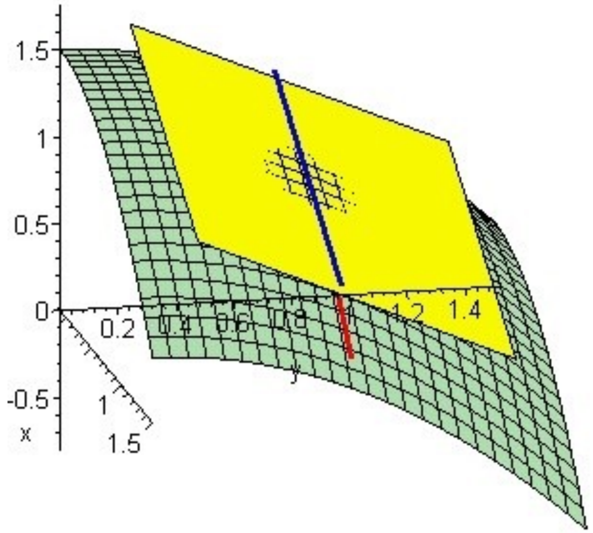
$$\nabla_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}^T$$

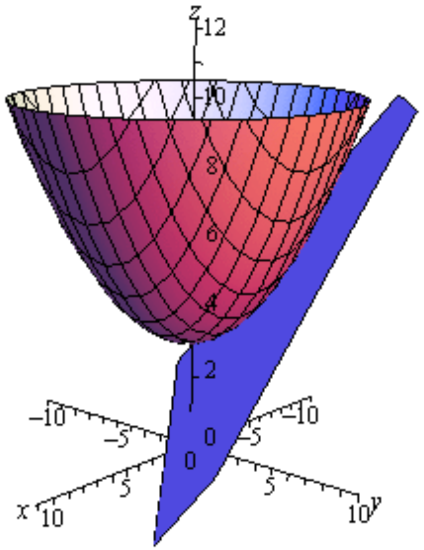
- Then, the **Jacobian matrix** is defined as

$$\mathbf{F}_{\mathbf{x}} = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

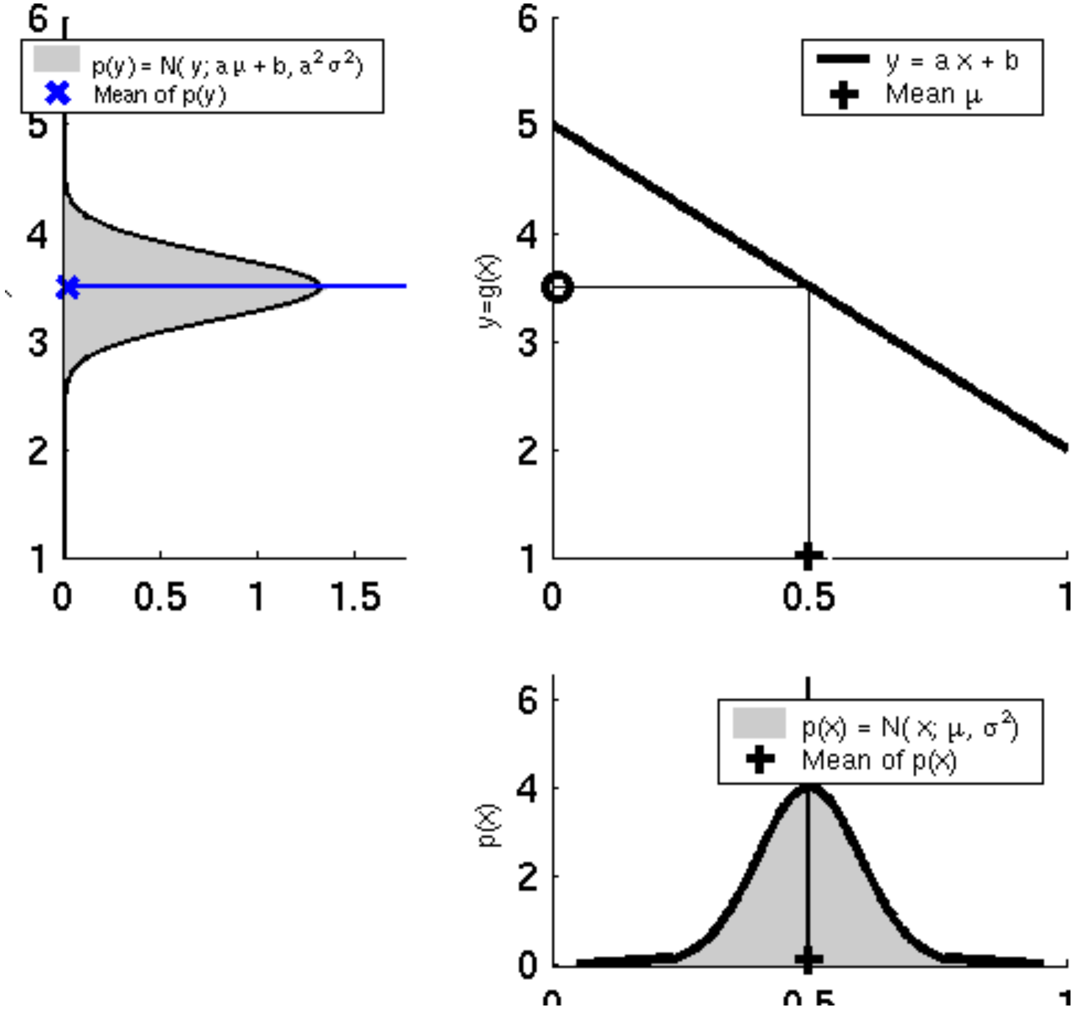
- It's the orientation of the **tangent plane** to the vector-valued function at a given point





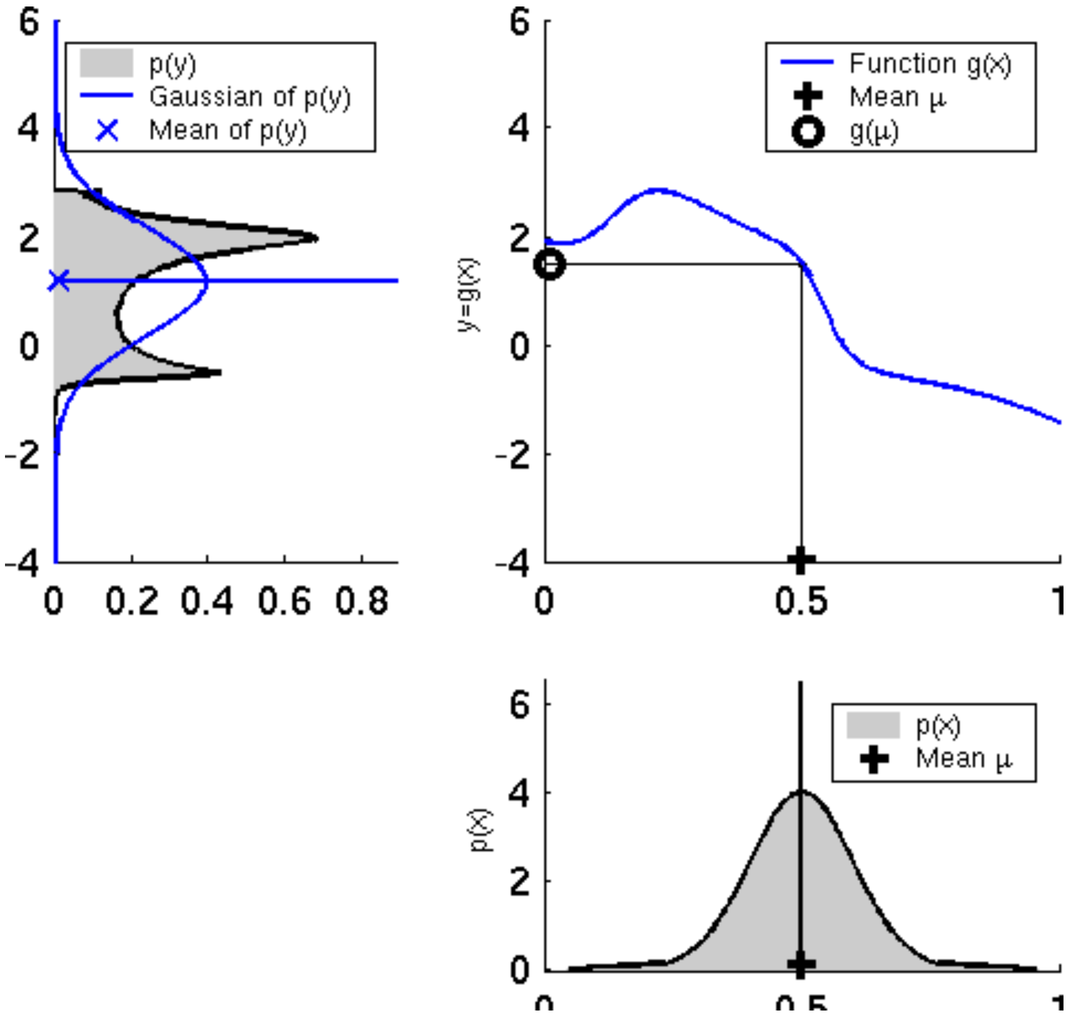
- **Generalizes the gradient** of a scalar valued function
- Heavily used for **first-order error propagation...**

Linearity Assumption Revisited



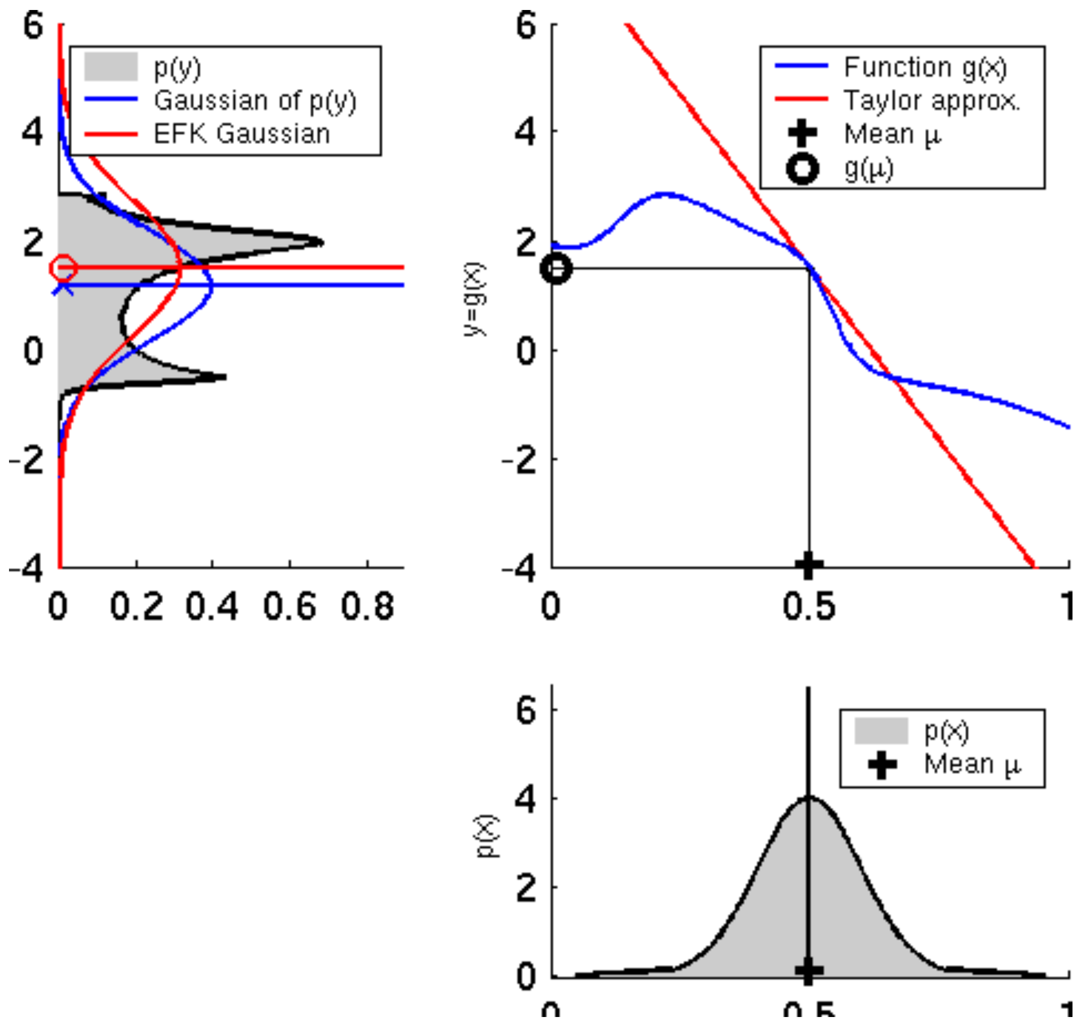
Courtesy: Thrun, Burgard, Fox

Non-Linear Function



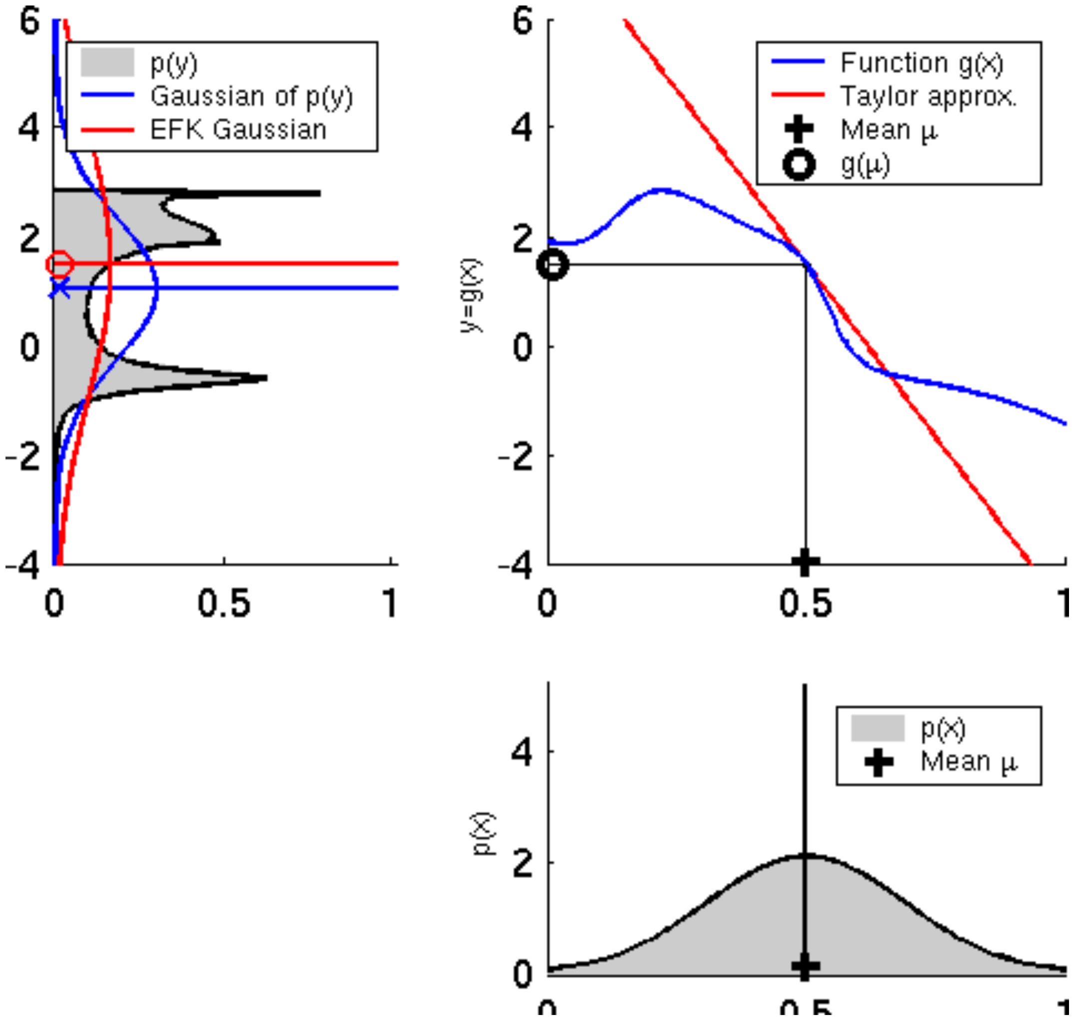
Courtesy: Thrun, Burgard, Fox

EKF Linearization (1)



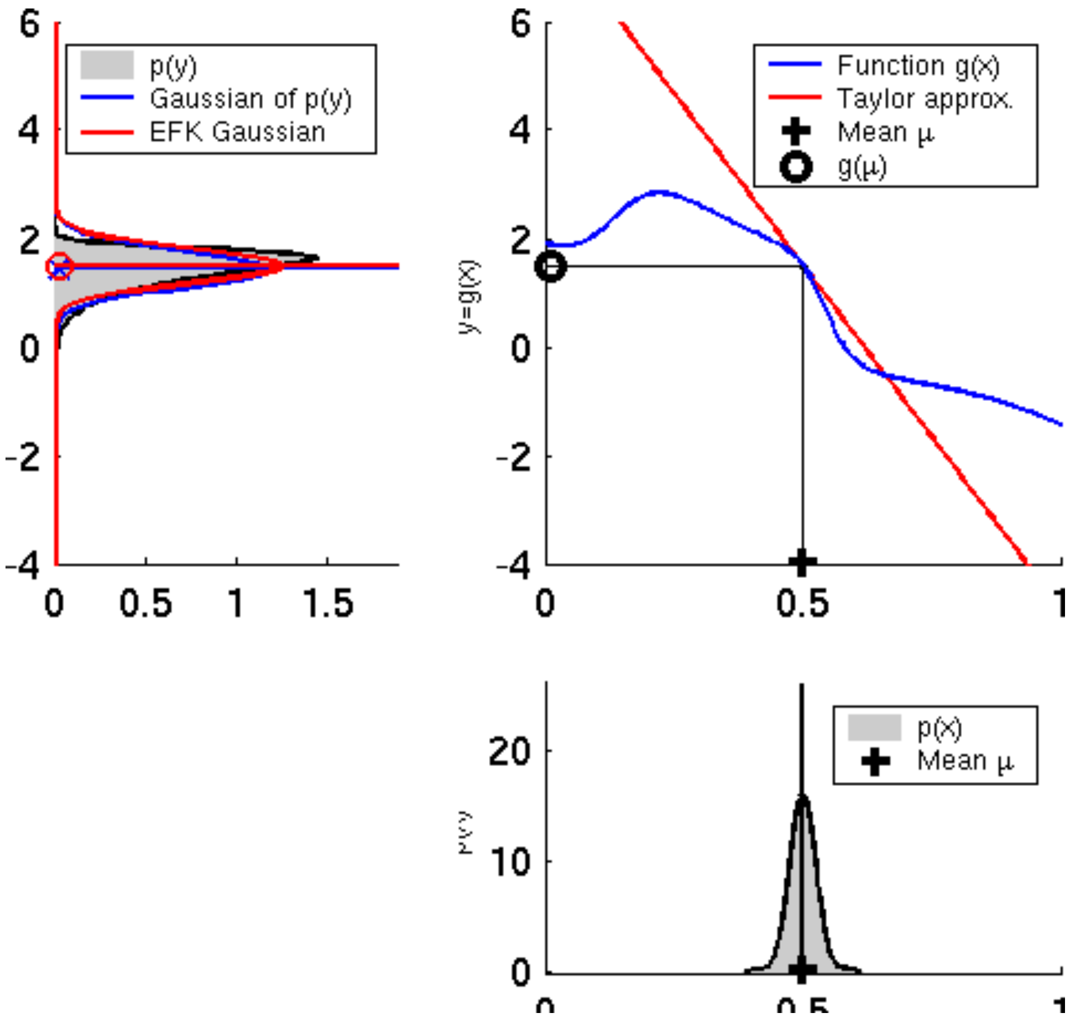
Courtesy: Thrun, Burgard, Fox

EKF Linearization (2)



Courtesy: Thrun, Burgard, Fox

EKF Linearization (3)



Courtesy: Thrun, Burgard, Fox

Linearized Motion Model

- The linearized model leads to

$$p(x_t \mid u_t, x_{t-1}) \approx \det\left(2\pi R_t\right)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2}\left(x_t - g(u_t, \mu_{t-1}) - G_t\,(x_{t-1} - \mu_{t-1})\right)^T\right.$$
$$\left. R_t^{-1}\left(x_t - \underbrace{g(u_t, \mu_{t-1}) - G_t\,(x_{t-1} - \mu_{t-1})}_{\text{linearized model}}\right)\right)$$

- R_t describes the noise of the motion

Linearized Observation Model

- The linearized model leads to

$$p(z_t \mid x_t) = \det\left(2\pi Q_t\right)^{-\frac{1}{2}}$$
$$\exp\left(-\frac{1}{2} \left(z_t - h(\bar{\mu}_t) - H_t \left(x_t - \bar{\mu}_t\right)\right)^T \right.$$
$$\left. Q_t^{-1} \left(z_t - \underbrace{h(\bar{\mu}_t) - H_t \left(x_t - \bar{\mu}_t\right)}_{\text{linearized model}}\right)\right)$$

- Q_t describes the measurement noise

Extended Kalman Filter Algorithm

1: **Extended_Kalman_filter**$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

2: $\bar{\mu}_t = \underline{g(u_t, \mu_{t-1})}$

3: $\bar{\Sigma}_t = G_t \, \Sigma_{t-1} \, G_t^T + R_t$

4: $K_t = \bar{\Sigma}_t \, H_t^T (H_t \, \bar{\Sigma}_t \, H_t^T + Q_t)^{-1}$

5: $\mu_t = \bar{\mu}_t + K_t(z_t - \underline{h(\bar{\mu}_t)})$

6: $\Sigma_t = (I - K_t \, H_t) \, \bar{\Sigma}_t$

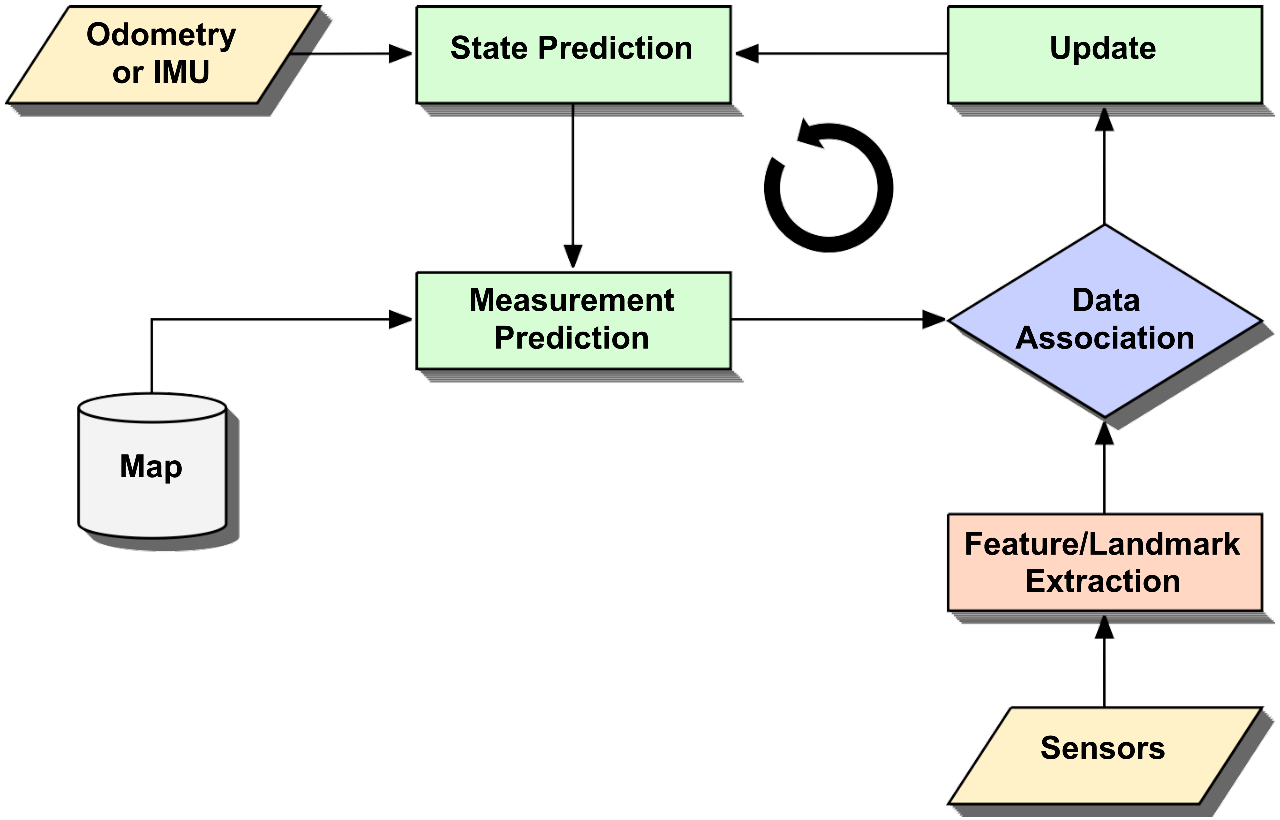
7: return μ_t, Σ_t

$$\boxed{\begin{array}{c} A_t \leftrightarrow G_t \\ C_t \leftrightarrow H_t \end{array}}$$

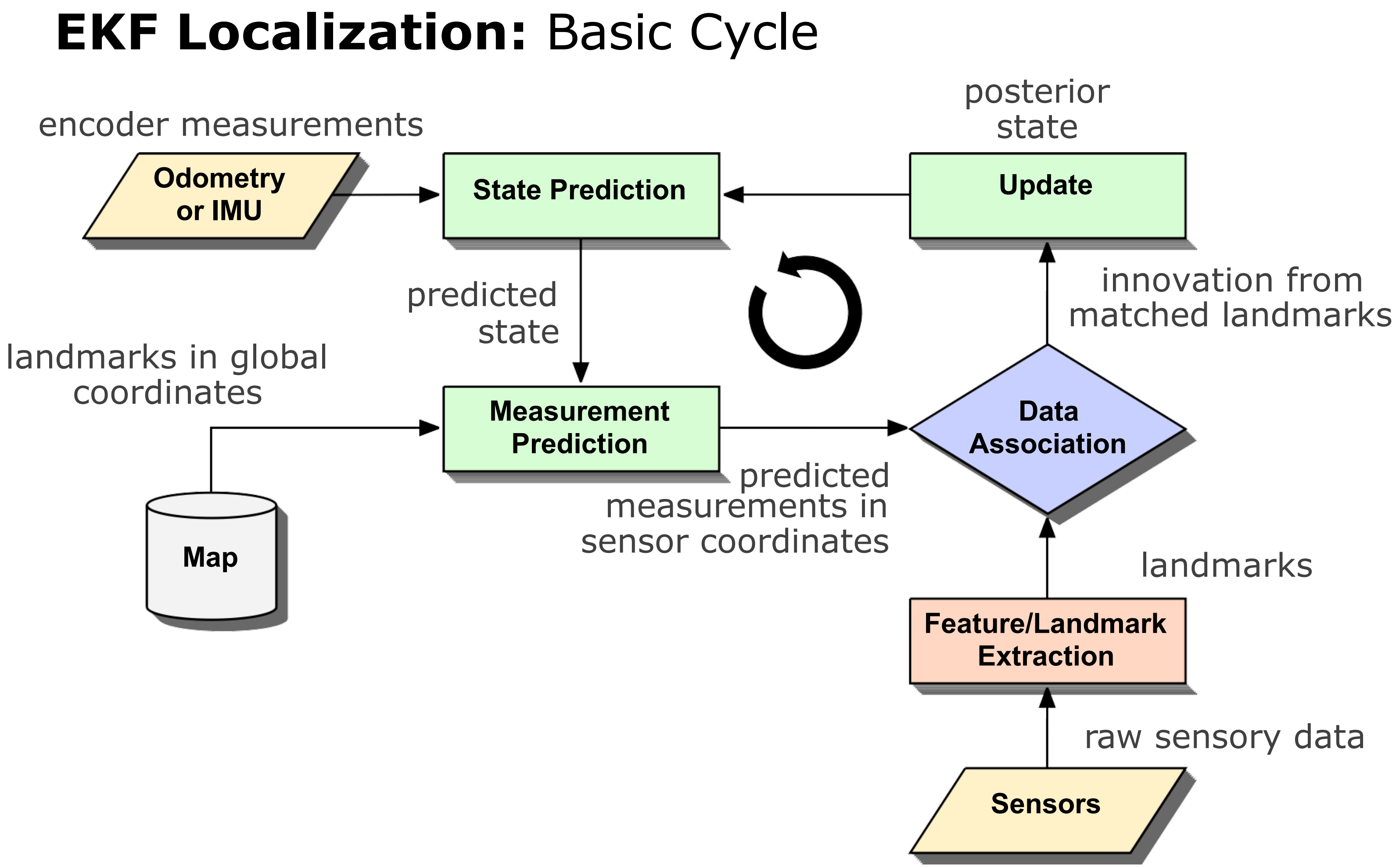
KF vs. EKF

Landmark-based Localization

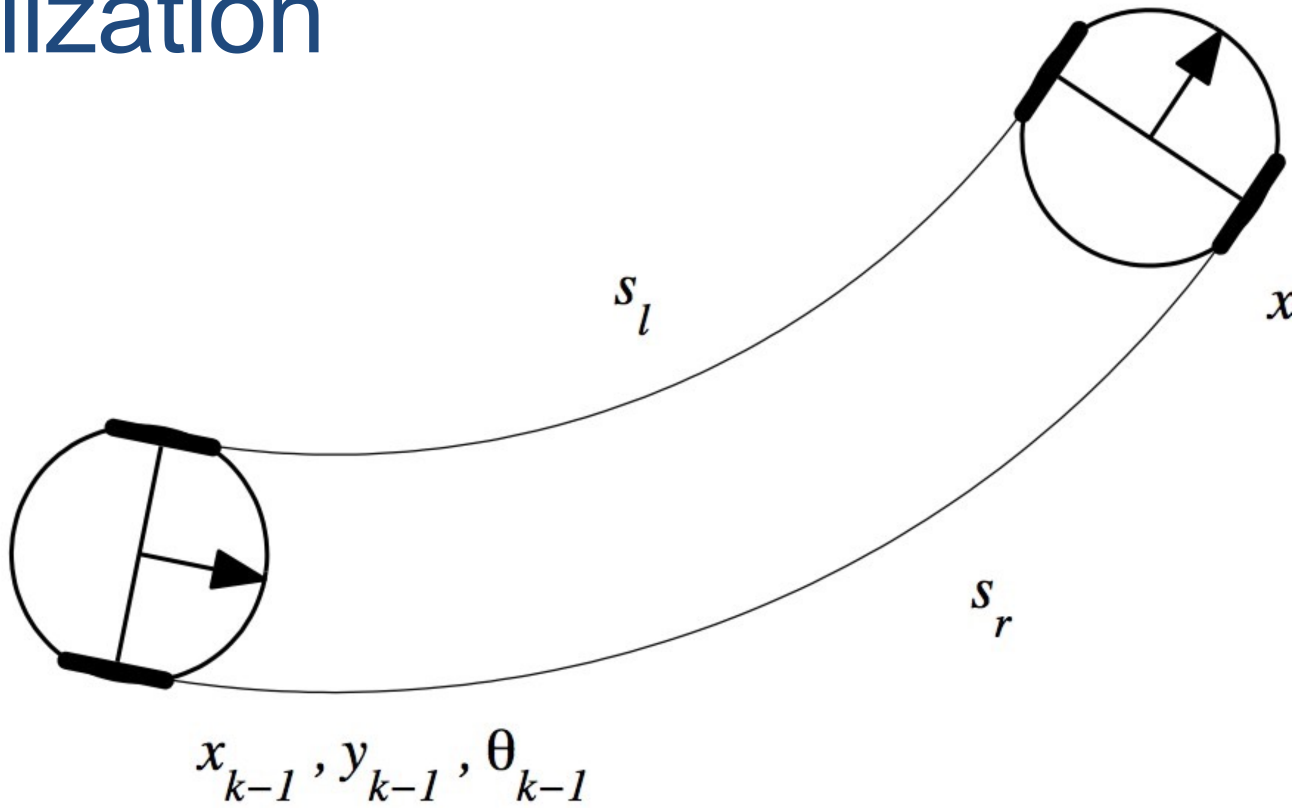
EKF Localization: Basic Cycle



Landmark-based Localization



Landmark-based Localization



State Prediction (Odometry)

$$\hat{\mathbf{x}}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k)$$
$$\hat{C}_k = F_x C_k F_x^T + F_u U_k F_u^T$$

Control $\mathbf{u}_k$: wheel displacements s_l, s_r

$$\mathbf{u}_k = (s_l \ s_r)^T \qquad U_k = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

Error model: linear growth

$$\begin{aligned} \sigma_l &= k_l \, |s_l| \\ \sigma_r &= k_r \, |s_r| \end{aligned}$$

Nonlinear process model f:

$$\mathbf{x}_k = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{b}{2} \frac{s_l+s_r}{s_r-s_l} \left(-\sin\theta_{k-1} + \sin(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{b}{2} \frac{s_l+s_r}{s_r-s_l} \left(\cos\theta_{k-1} - \cos(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{s_r-s_l}{b} \end{bmatrix}$$

Landmark-based Localization

State Prediction (Odometry)

$$\hat{\mathbf{x}}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k)$$

$$\hat{C}_k = F_x\, C_k\, F_x^T + F_u\, U_k\, F_u^T$$

Control $\mathbf{u}_k$: wheel displacements s_l, s_r

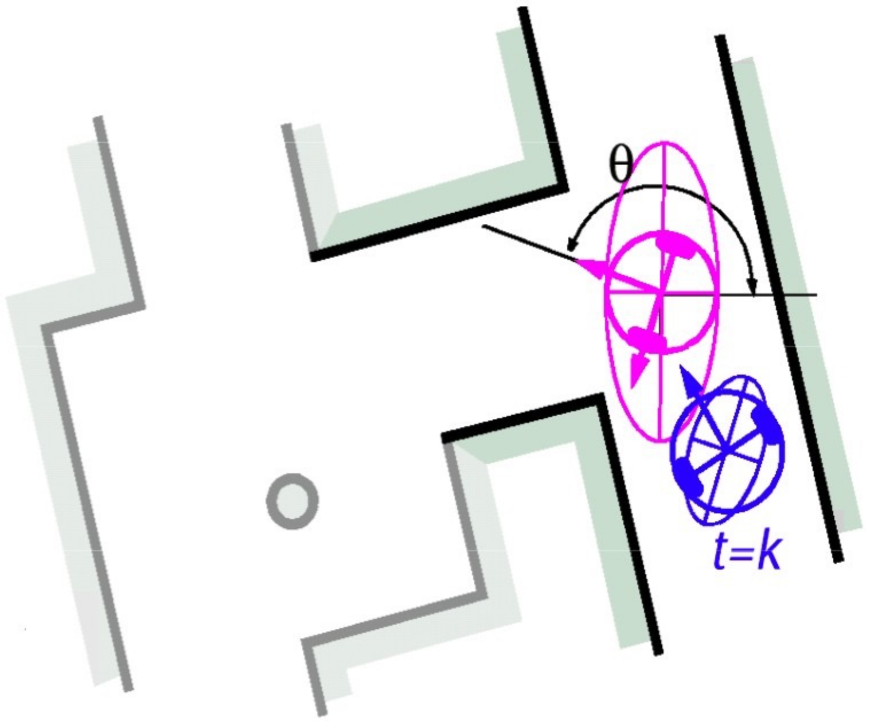
$$\mathbf{u}_k = (s_l \;\; s_r)^T \qquad U_k = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

Error model: linear growth

$$\begin{aligned} \sigma_l &= k_l\,|s_l| \\ \sigma_r &= k_r\,|s_r| \end{aligned}$$

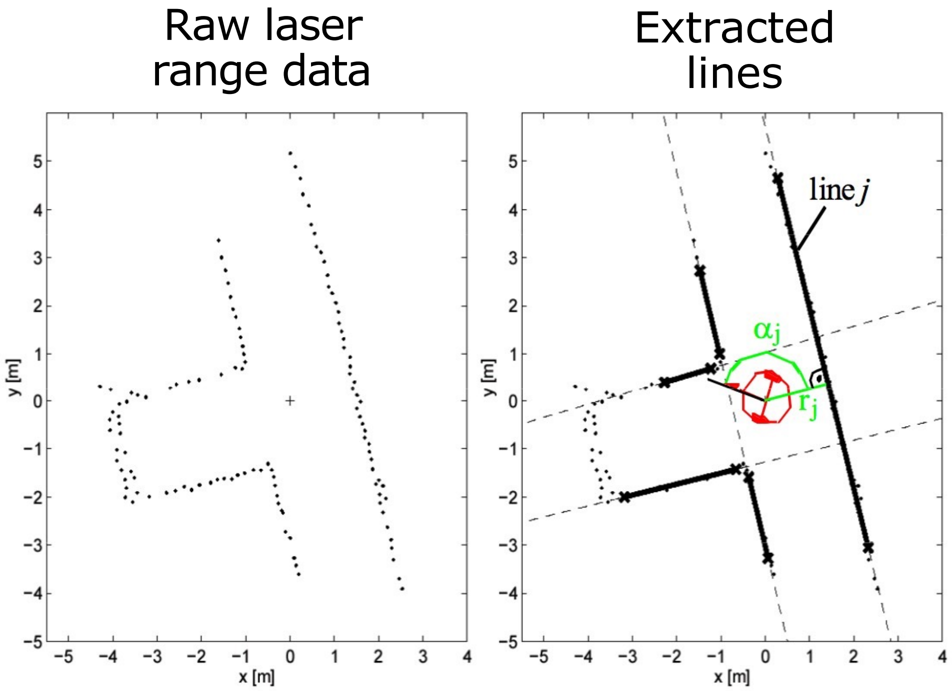
Nonlinear process model f:

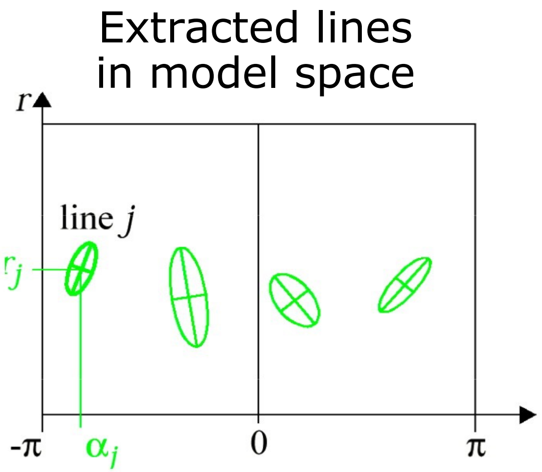
$$\mathbf{x}_k = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(-\sin\theta_{k-1} + \sin(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(\cos\theta_{k-1} - \cos(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{s_r-s_l}{b} \end{bmatrix}$$



Landmark-based Localization

Landmark Extraction (Observation)





Hessian line model

$$x\ cos(\alpha) + y\ sin(\alpha) - r = 0$$

$$\mathbf{z}_k = \begin{bmatrix} \alpha \\ r \end{bmatrix}$$

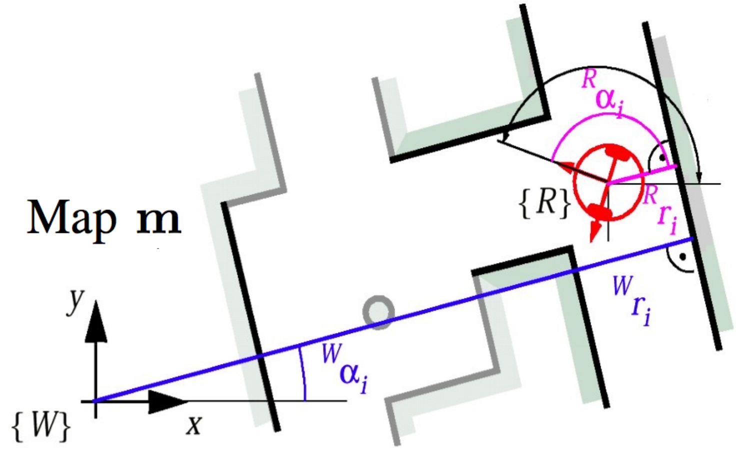
$$R_k = \begin{bmatrix} \sigma_\alpha^2 & \sigma_{\alpha r} \\ \sigma_{r\alpha} & \sigma_r^2 \end{bmatrix}$$

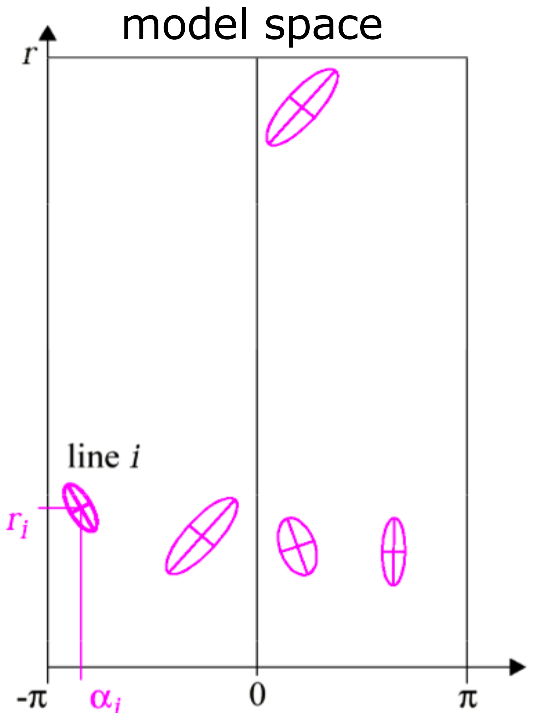
Landmark-based Localization

Measurement Prediction

- ...is a coordinate frame transform world-to-sensor
- Given the predicted state (robot pose), predicts the location $\hat{\mathbf{z}}_k$ and location uncertainty $H\,\hat{C}_k\,H^T$ of expected observations in sensor coordinates

$$\hat{\mathbf{z}}_k = h(\hat{\mathbf{x}}_k, \mathbf{m})$$





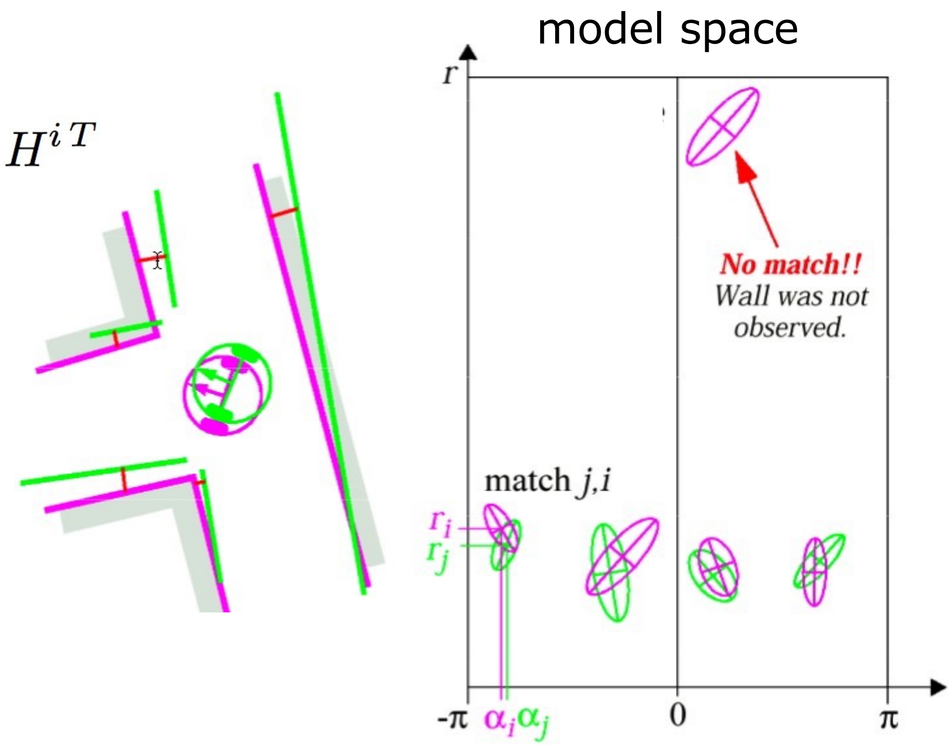
Landmark-based Localization

Data Association (Matching)

- Associates predicted measurements $\hat{\mathbf{z}}_k^i$ with observations $\mathbf{z}_k^j$

$$\nu_k^{ij} = \mathbf{z}_k^j - \hat{\mathbf{z}}_k^i$$
$$S_k^{ij} = R_k^j + H^i \, \hat{C}_k \, H^{i\,T}$$

- Innovation and innovation covariance ν_k^{ij}



Green: observation

Magenta: measurement prediction

Landmark-based Localization

Update

- Kalman gain

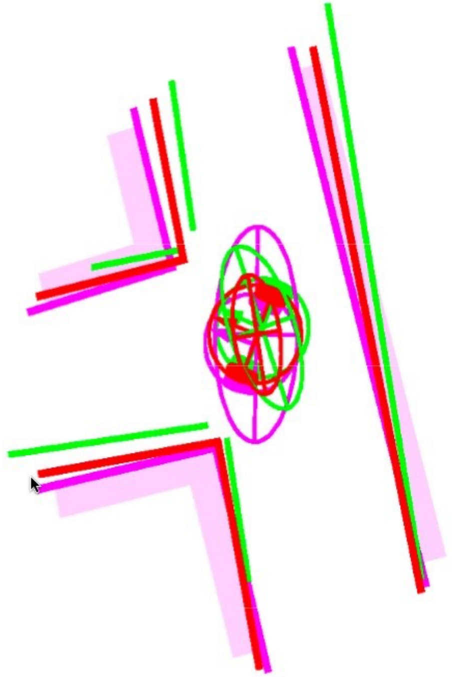
$$K_k = \hat{C}_k \, H^T S_k^{-1}$$

- State update (robot pose)

$$\mathbf{x}_k = \hat{\mathbf{x}}_k + K_k \, \nu_k$$

- State covariance update

$$C_k = (I - K_k \, H) \, \hat{C}_k$$



Red: posterior estimate

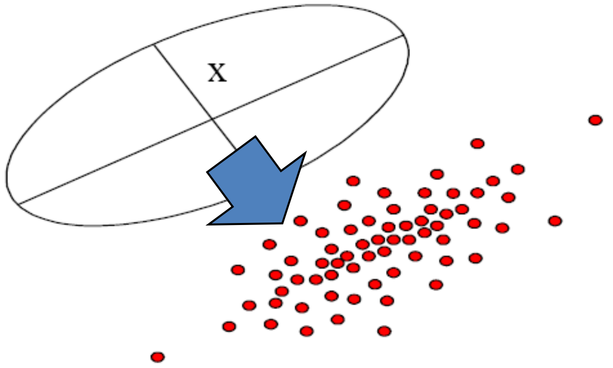
PARTICLE FILTER

Following Material:

- Wolfram Burgard, University of Freiburg

Particle Filter SLAM: FastSLAM

- **FastSLAM approach**
  - Using particle filters.
  - Particle filters: mathematical models that represent probability distribution as a set of discrete particles that occupy the state space.



probability distribution (ellipse) as particle set (red dots)

- Particle filter update
  - Generate new particle distribution using motion model and controls

a) For each particle:

1. Compare particle's prediction of measurements with actual measurements
2. Particles whose predictions match the measurements are given a high weight

b) Filter resample:

- Resample particles based on weight
- Filter resample
  - Assign each particle a weight depending on how well its estimate of the state agrees with the measurements and randomly draw particles from previous distribution based on weights creating a new distribution.

Motivation

- Particle filters are a way to efficiently represent non-Gaussian distribution
- Basic principle
  - Set of state hypotheses (“particles”)
  - Survival-of-the-fittest

Mathematical Description

- Set of weighted samples

$$S = \{< s^{[i]}, w^{[i]} > | \ i = 1, \dots, N\}$$

State hypothesis

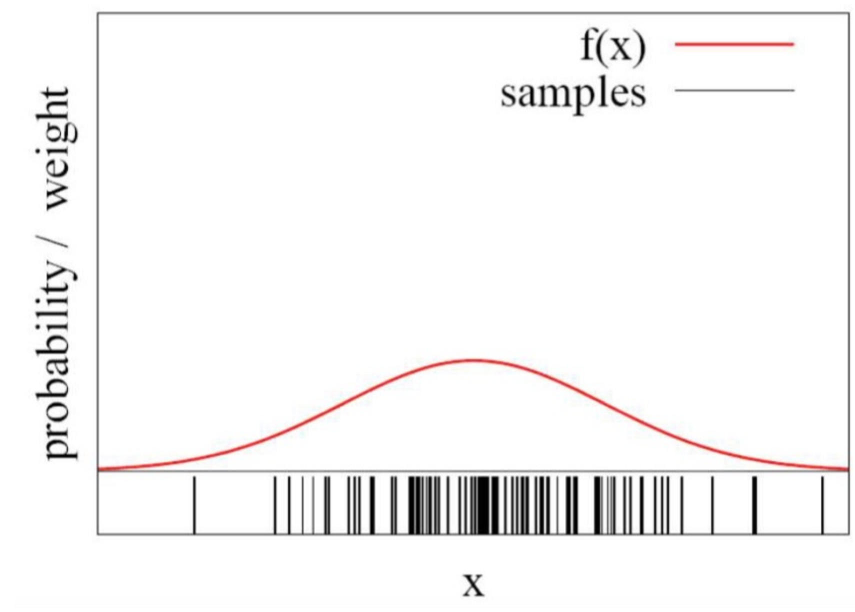
Importance Weight

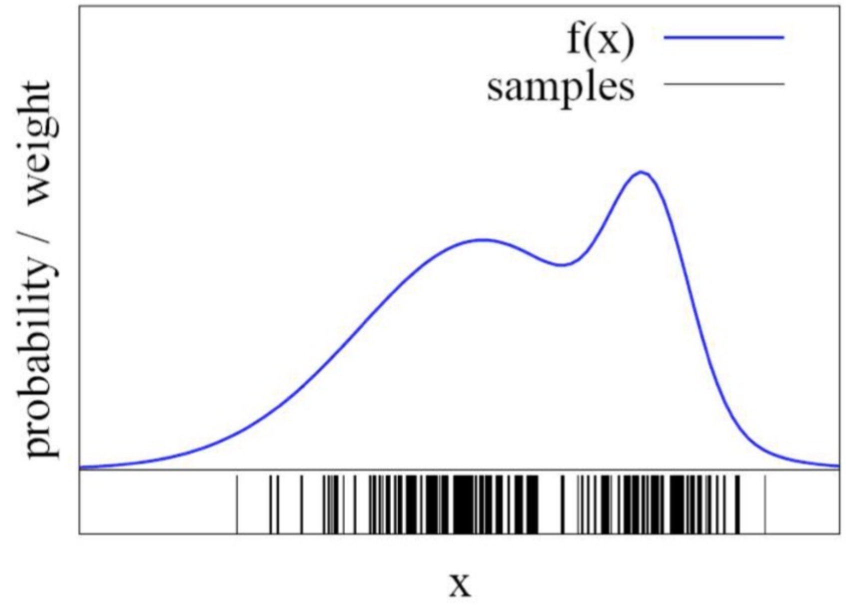
- The samples represent the posterior

$$p(x) = \sum_{i=1}^{N} w_i \cdot \delta_{s^{[i]}}(x)$$

Function Approximation

- Particle sets can be used to approximate functions

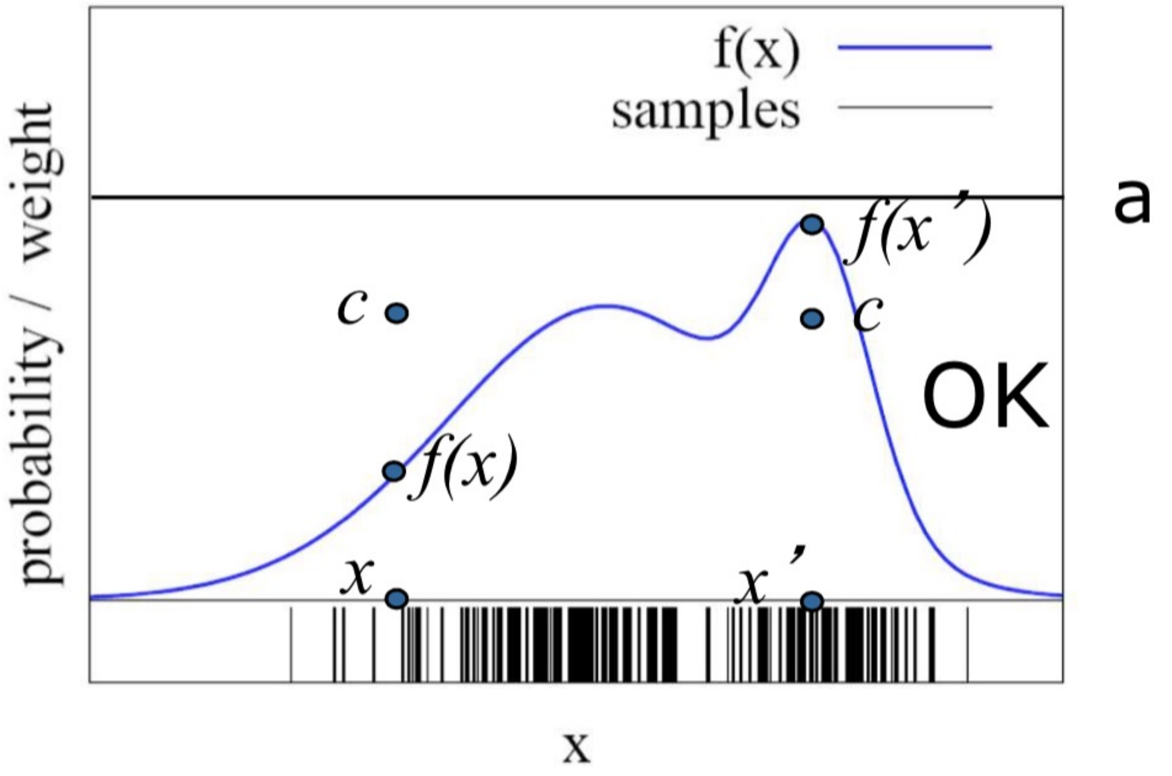




- The more particles fall into an interval, the higher the probability of that interval
- How to draw samples from a function/distribution?

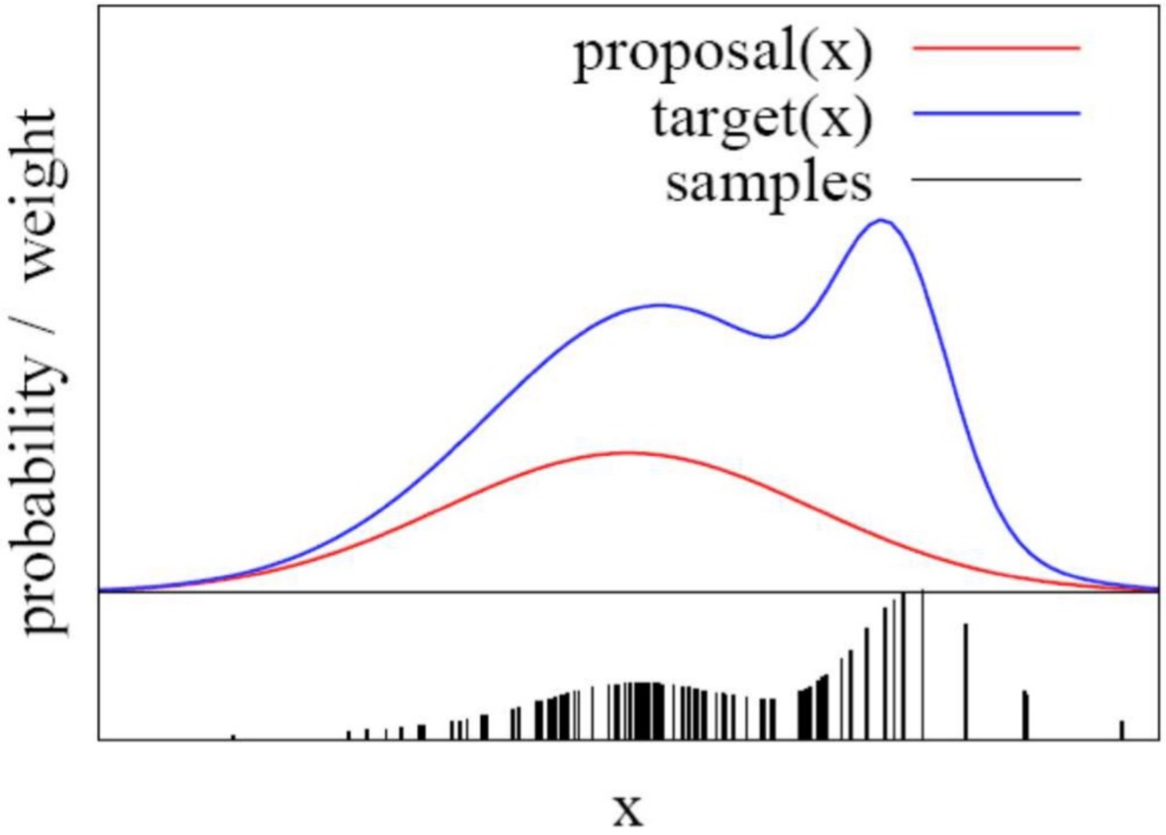
Rejection Sampling

- Let us assume that $f(x) < a$ for all x
- Sample x from a uniform distribution
- Sample c from $[0, a]$
- if $f(x) > c$ keep the sample
- otherwise reject the sample

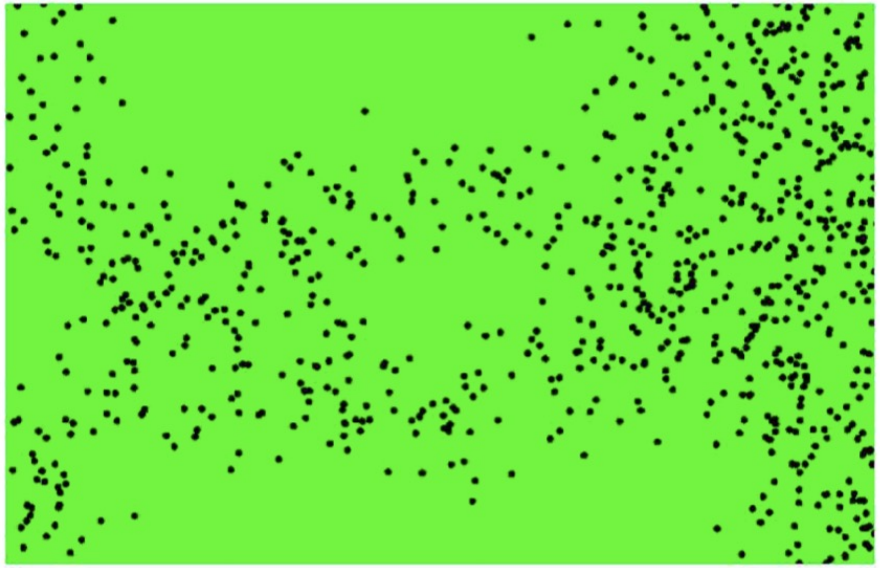


Importance Sampling Principle

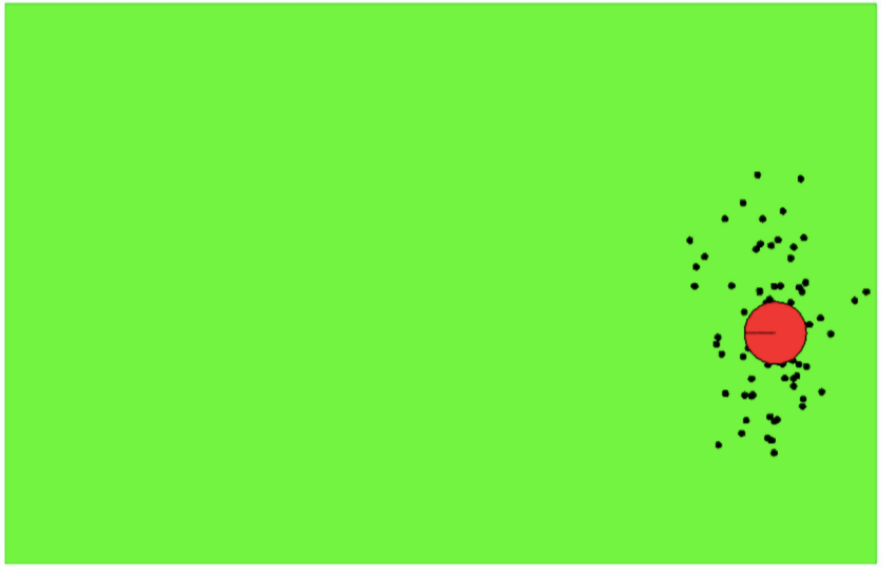
- We can even use a different distribution g to generate samples from f
- By introducing an importance weight w, we can account for the “differences between g and f ”
- $w = f/g$
- f is called target
- g is called proposal
- Pre-condition:
  - $f(x) > 0 \rightarrow g(x) > 0$



Importance Sampling with Resampling



Weighted Samples



After Resampling

Particle Filter Algorithm

- Sample the next generation for particles using the proposal distribution
- Compute the importance weights :

$$weight = target\ distribution\ /\ proposal\ distribution$$

- Resampling: “Replace unlikely samples by more likely ones”

Particle Filter Algorithm

1. Algorithm particle_filter(S_{t-1}, u_t, z_t):
2. $S_t = \emptyset, \eta = 0$
3. For $i = 1, \dots, n$ — Generate new samples
4. Sample index $j(i)$ from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t | x_{t-1}, u_t)$ using $x_{t-1}^{j(i)}$ and u_t
6. $w_t^i = p(z_t | x_t^i)$ — Compute importance weight
7. $\eta = \eta + w_t^i$ — Update normalization factor
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ — Add to new particle set
9. For $i = 1, \dots, n$
10. $w_t^i = w_t^i / \eta$ — Normalize weights

Particle Filter Algorithm

- $bel(x_t) = \eta p(z_t|x_t) \int p(x_t|x_{t-1}, u_t) bel(x_{t-1}) dx_{t-1}$

Draw x_{t-1}^i from $bel(x_{t-1})$

Draw x_t^i from $p(x_t|x_{t-1}, u_t)$

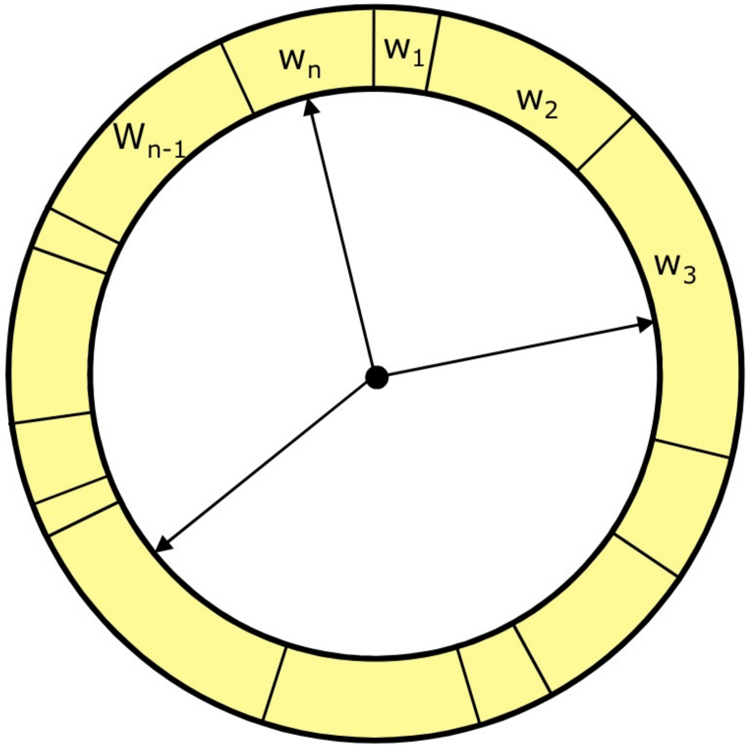
Importance factor for x_t^i

$$w_t^i = \frac{target\ distribution}{proposal\ distribution} = \frac{\eta p(z_t|x_t) p(x_t|x_{t-1}, u_t) bel(x_{t-1})}{p(x_t|x_{t-1}, u_t) bel(x_{t-1})} \propto \eta p(z_t|x_t)$$

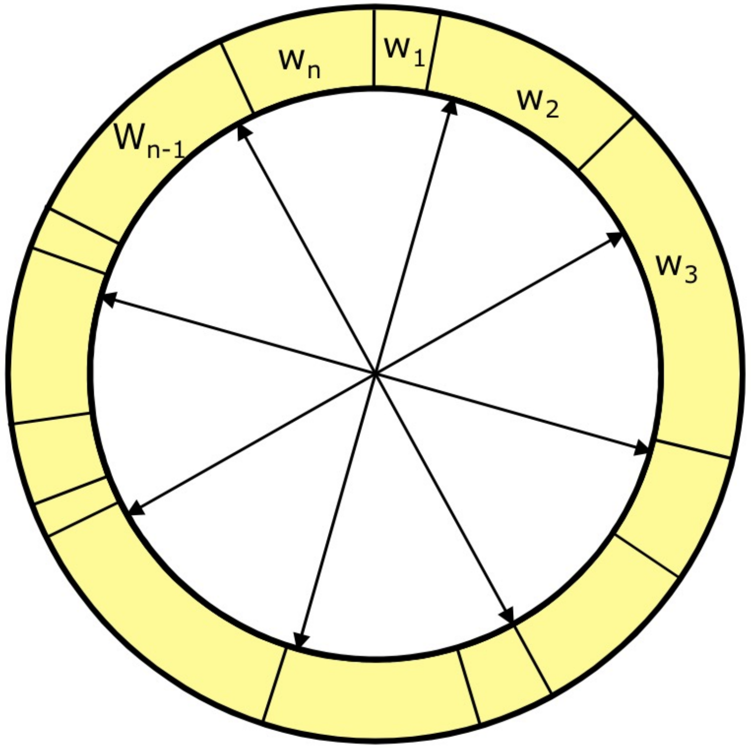
Resampling

- Given: Set S of weighted samples.
- Wanted : Random sample, where the probability of drawing x_i is given by w_i.
- Typically done n times with replacement to generate new sample set S'.

Resampling



- Roulette wheel
- Binary search, $n \log n$



- Stochastic universal sampling
- Systematic resampling
- Linear time complexity
- Easy to implement, low variance

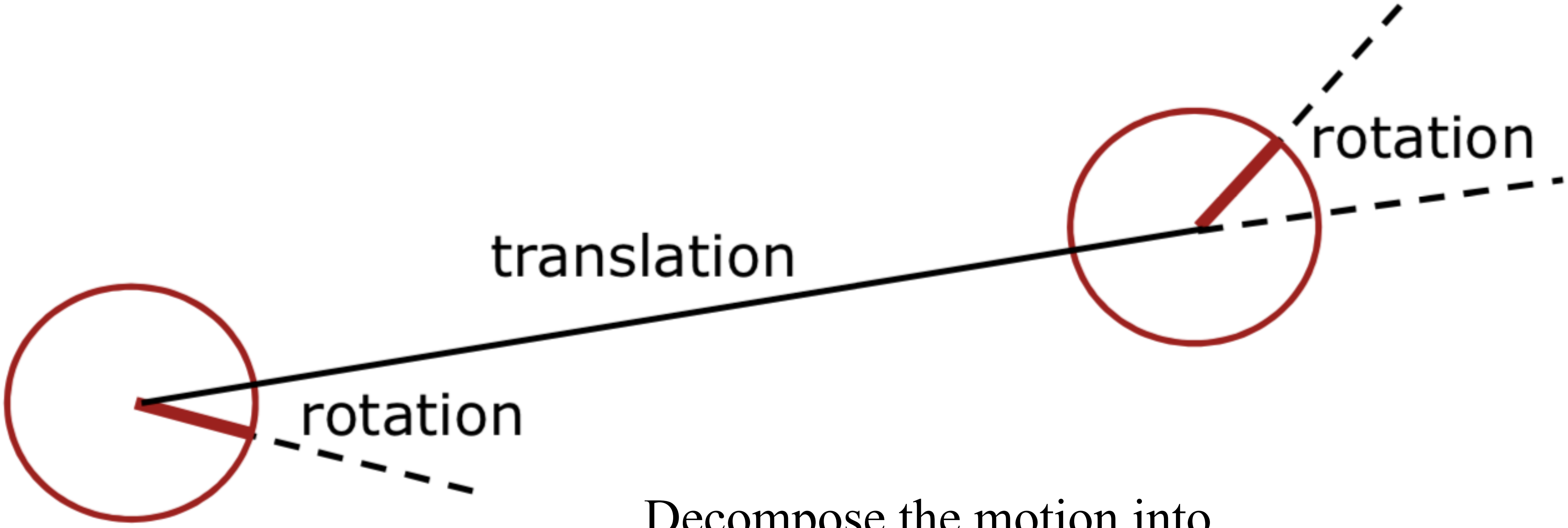
Mobile Robot Localization

- Each particle is a potential pose of the robot
- Proposal distribution is the motion model of the robot (prediction step)
- The observation model is used to compute the importance weight (correction step)

Motion Model Reminder

According to the estimated motion

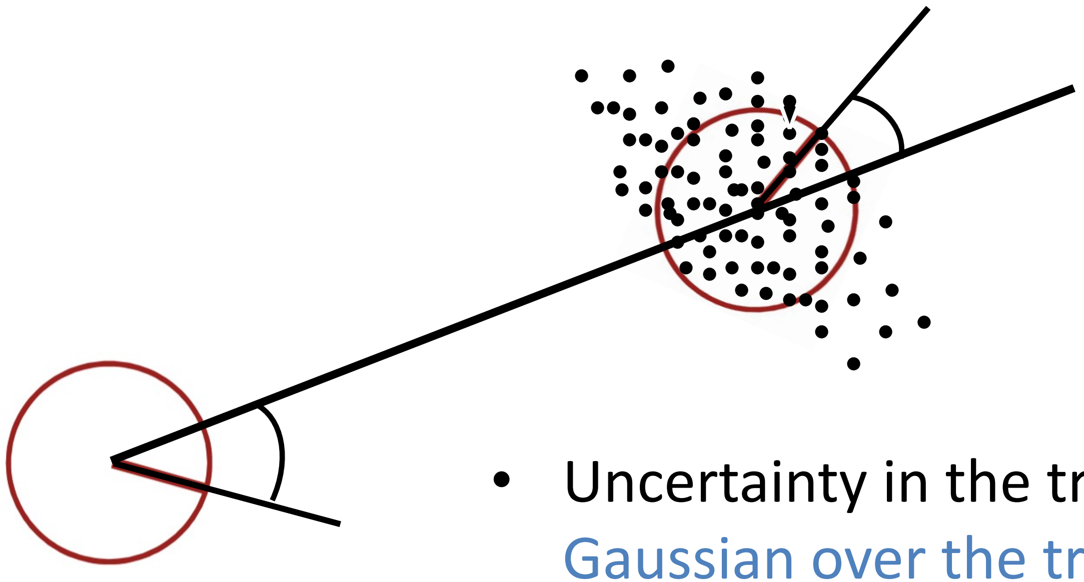
Motion Model Reminder



Decompose the motion into

- Traveled distance
- Start rotation
- End rotation

Motion Model Reminder



- Uncertainty in the translation of the robot: Gaussian over the traveled distance
- Uncertainty in the rotation of the robot: Gaussians over start and end rotation
- For each particle, draw a new pose by sampling from these three individual normal distributions

Mobile Robot Localization Using Particle Filters (1)

- Each particle is a potential pose of the robot
- The set of weighted particles approximates the posterior belief about the robot's pose (target distribution)

Mobile Robot Localization Using Particle Filters (2)

- Particles are drawn from the motion model (proposal distribution)
- Particles are weighted according to the observation model (sensor model)
- Particles are resampled according to the particle weights

Mobile Robot Localization Using Particle Filters (3)

- Why is resampling needed?
  - We only have a finite number of particles
  - Without resampling: The filter is likely to loose track of the "good" hypotheses
  - Resampling ensures that particles stay in the meaningful area of the state space

SLAM Using Particle Filters – Grid-based SLAM

- Can we solve the SLAM problem if no pre- defined landmarks are available?
- Can we use the ideas of FastSLAM to build grid maps?
- As with landmarks, the map depends on the poses of the robot during data acquisition
- If the poses are known, grid-based mapping is easy ("mapping with known poses")

Rao-Blackwellization

- $$\underbrace{p(\overset{\text{Poses}}{x_{1:t}}, \overset{\text{Map}}{m} | \overset{\text{Observations}}{z_{1:t}}, \overset{\text{Movements}}{u_{0:t-1}})}_{\text{SLAM posterior}} = \underbrace{p(x_{1:t}, |z_{1:t}, u_{0:t-1})}_{\text{Robot path posterior}} \cdot \underbrace{p(m|x_{1:t}, z_{1:t})}_{\text{Mapping with known poses}}$$

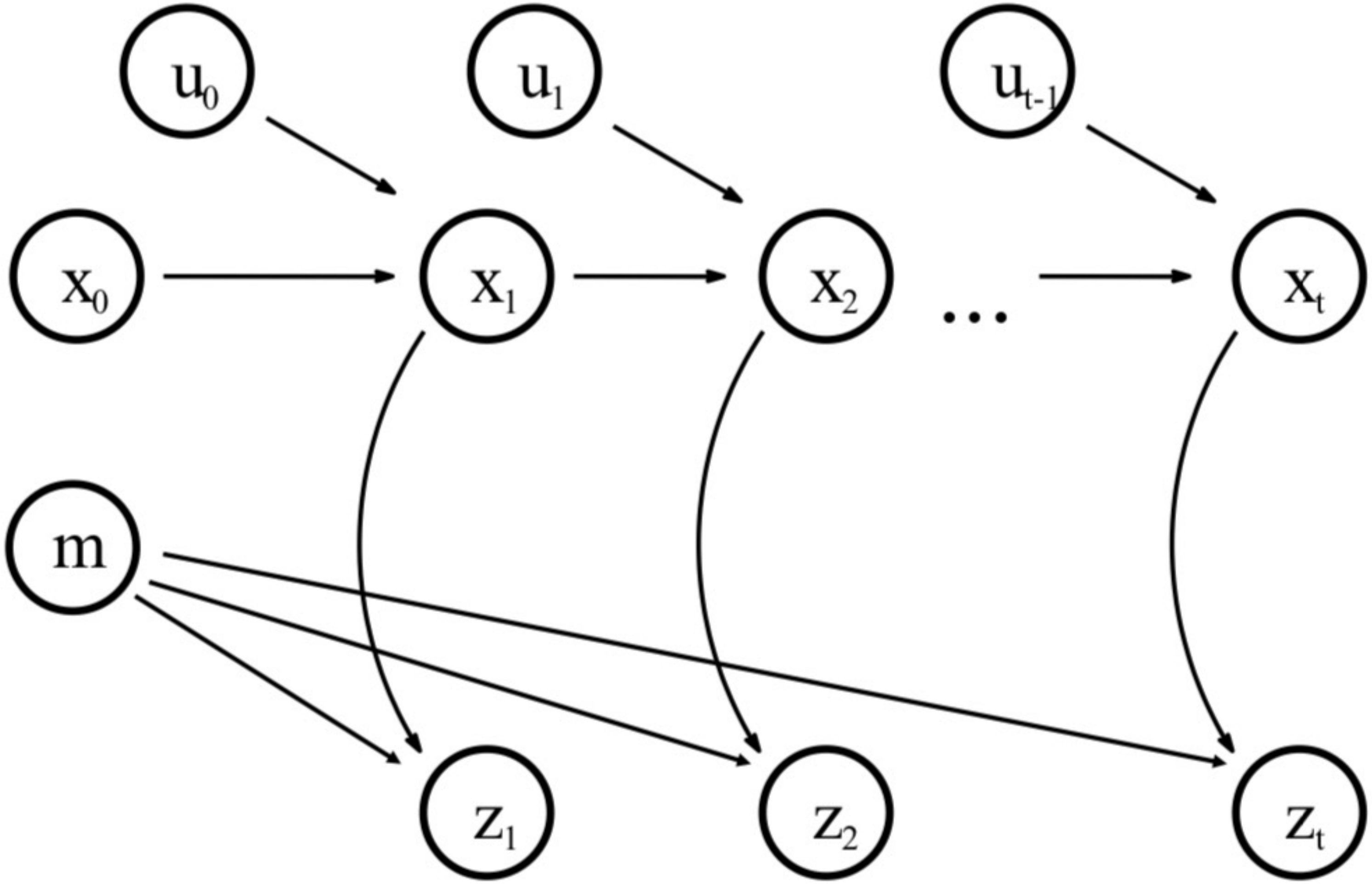
Rao-Blackwellization

- $p(x_{1:t}, m|z_{1:t}, u_{0:t-1}) = \underbrace{p(x_{1:t}, |z_{1:t}, u_{0:t-1})}_{\text{This is Localization, use MCL}} \cdot \underbrace{p(m|x_{1:t}, z_{1:t})}_{\text{Use the pose estimate from the MCL and apply mapping with known poses}}$

This is Localization, use MCL

Use the pose estimate from the MCL and apply mapping with known poses

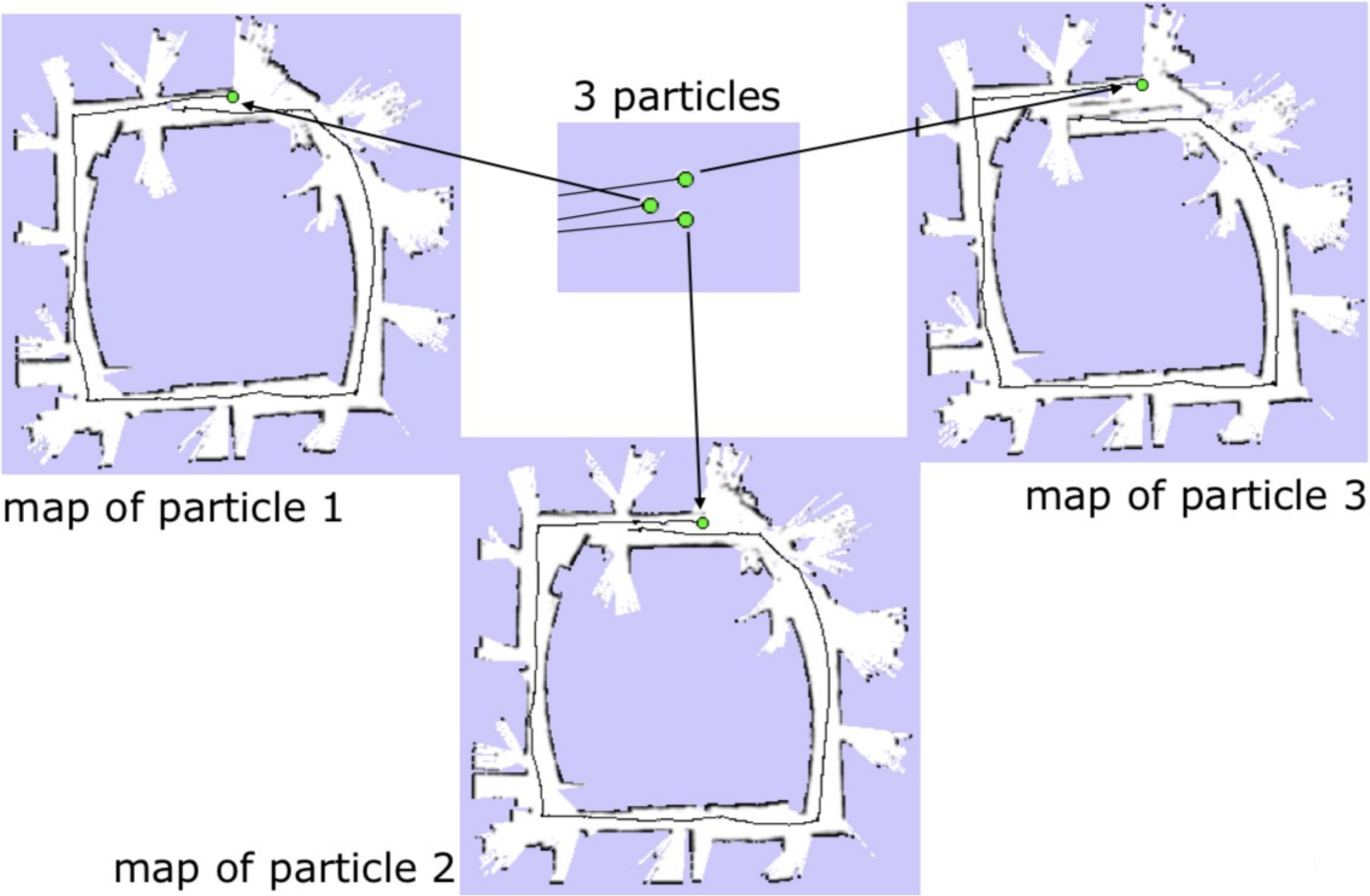
A Graphical Model of Mapping with Rao-Blackwellized PFs



Mapping with Rao- Blackwellized Particle Filters

- Each particle represents a possible trajectory of the robot
- Each particle
  - maintains its own map and
  - updates it upon “mapping with known poses”
- Each particle survives with a probability proportional to the likelihood of the observations relative to its own map

Particle Filter Example



Problem

- Each map is quite big in case of grid maps
- Each particle maintains its own map, therefore, one needs to keep the number of particles small
- **Solution:**
  Compute better proposal distributions!
- **Idea:**
  Improve the pose estimate before applying the particle filter

Pose Correction Using Scan Matching

- Maximize the likelihood of the i-th pose and map relative to the $(i-1)$-th pose and map

$$\hat{x}_t = argmax_{x_t}\{p(z_t|x_t, \widehat{m}_{t-1}) \cdot p(x_t|u_{t-1}, \hat{x}_{t-1})\}$$

current measurement

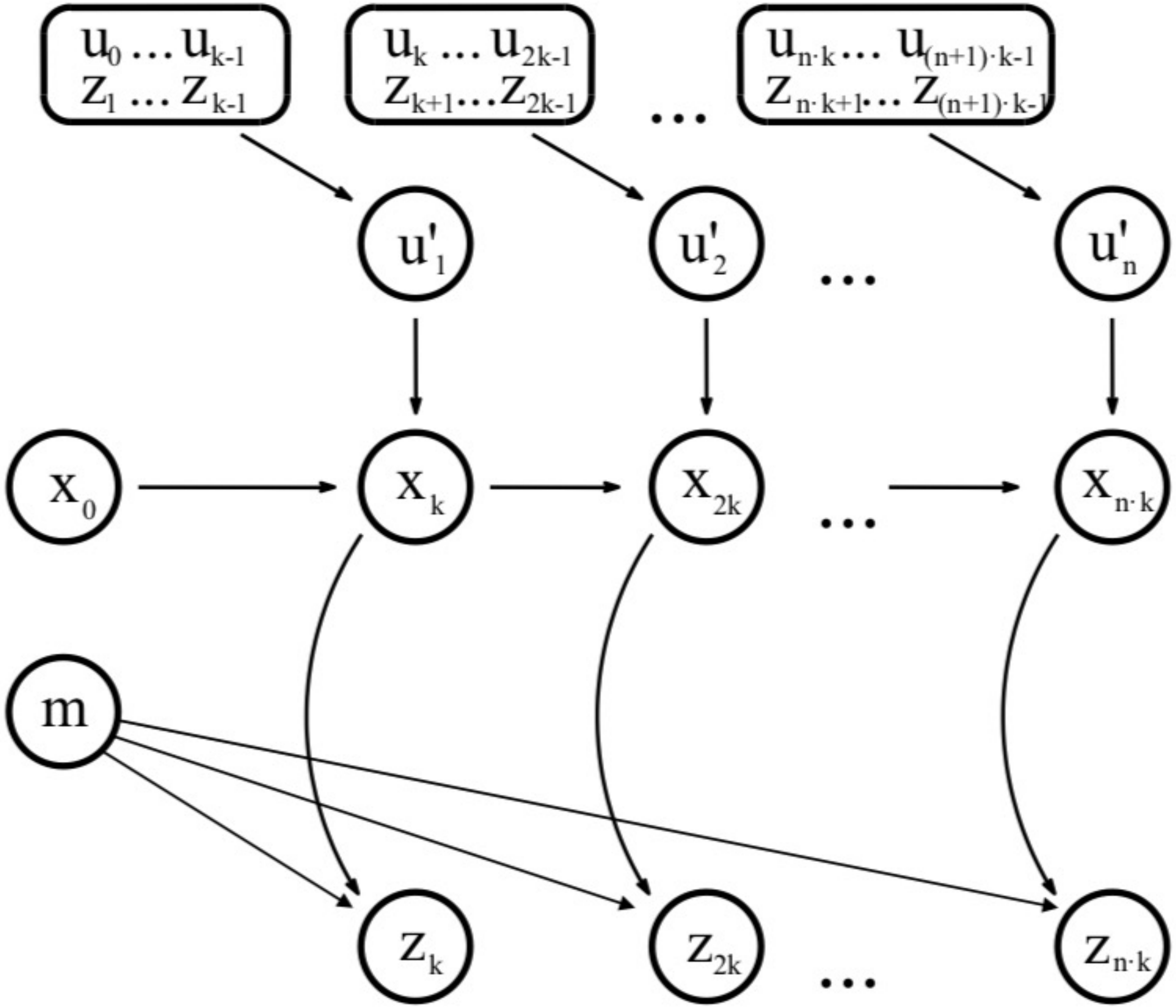
robot motion

map constructed so far

FastSLAM with Improved Odometry

- Scan-matching provides a locally consistent pose correction
- Pre-correct short odometry sequences using scan-matching and use them as input to FastSLAM
- Fewer particles are needed, since the error in the input is smaller

Graphical Model for Mapping with Improved Odometry

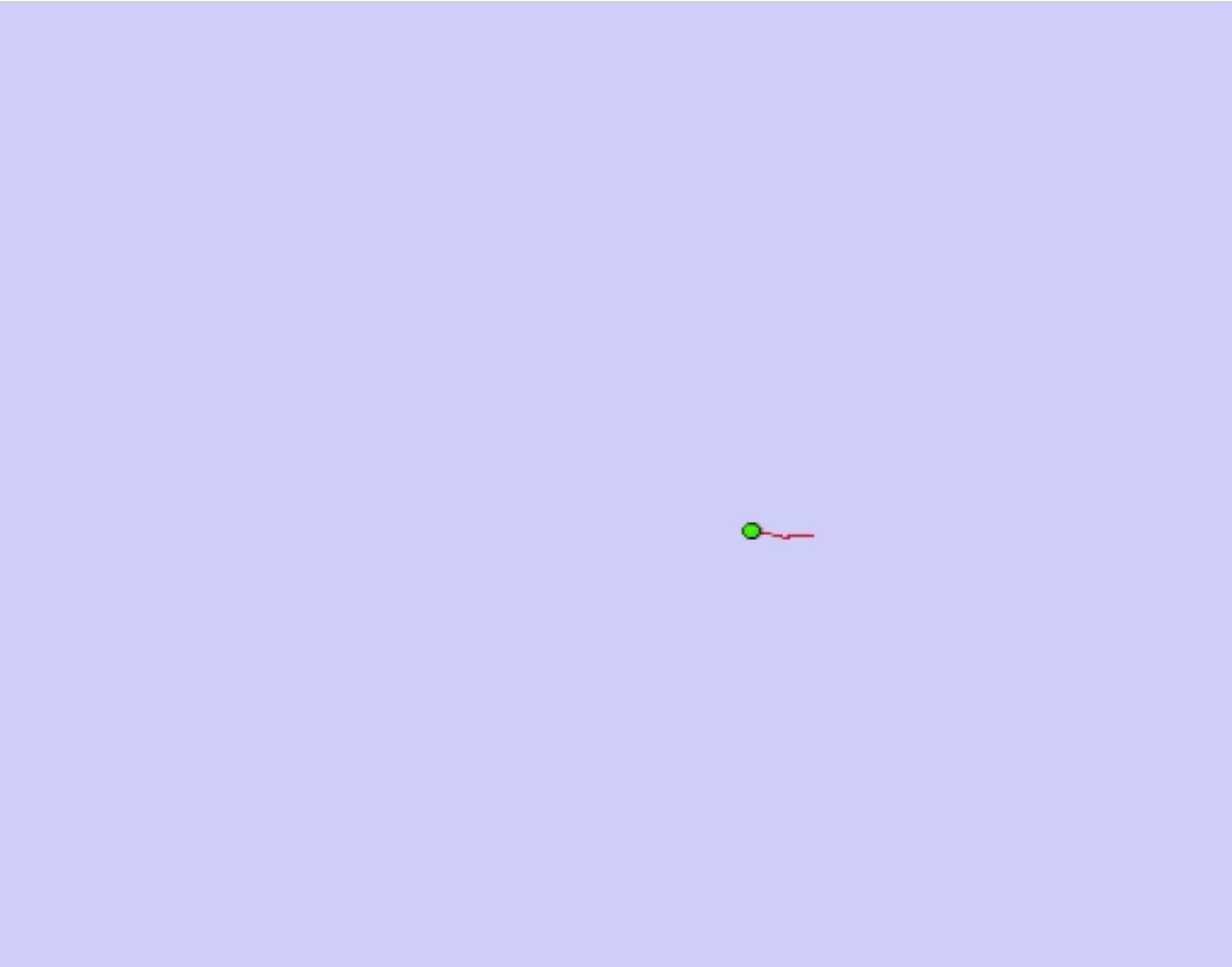


Raw Odometry

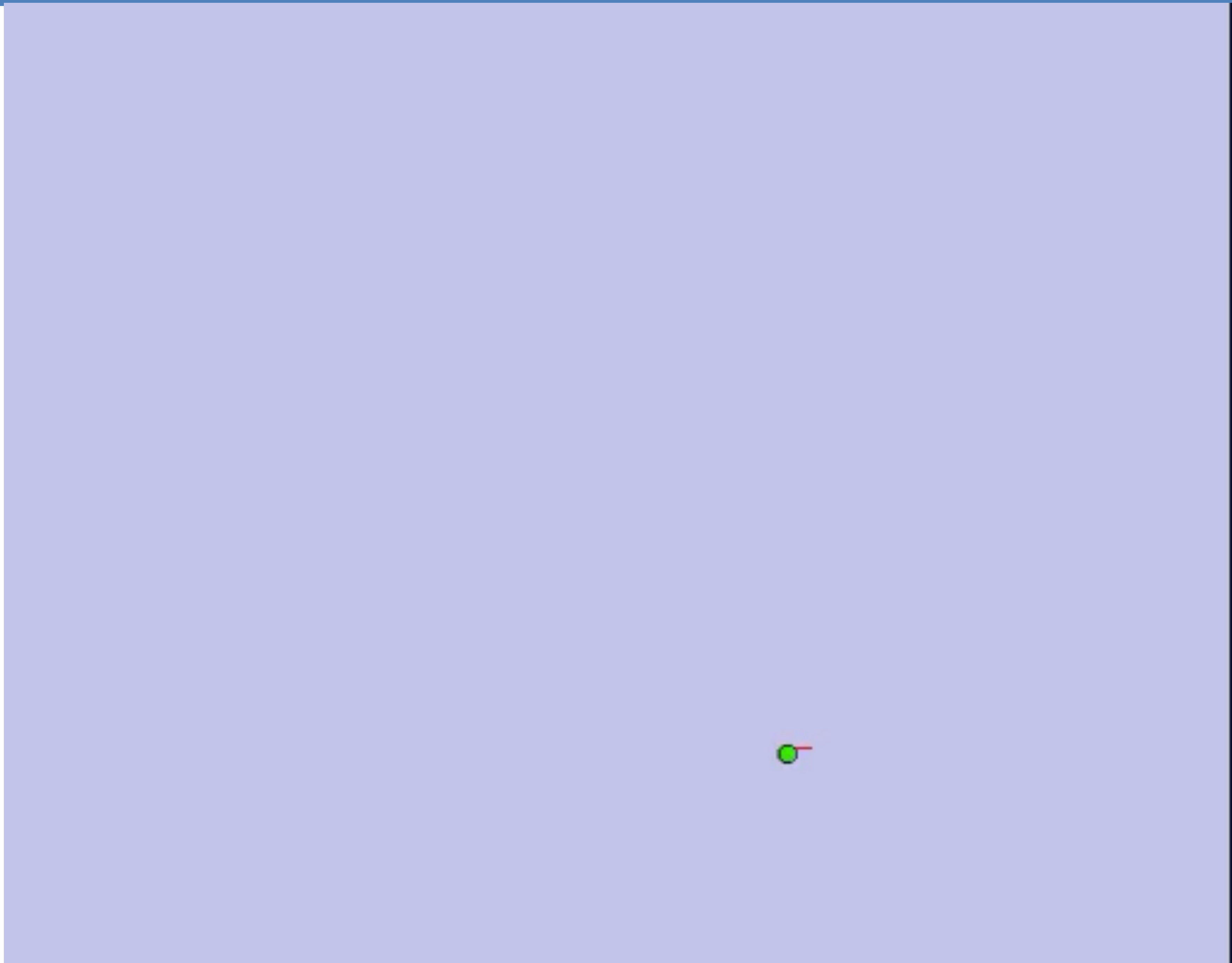
- Famous Intel Research Lab dataset (Seattle) by Dirk Hähnel

Courtesy of S. Thrun

http://robots.stanford.edu/videos.html

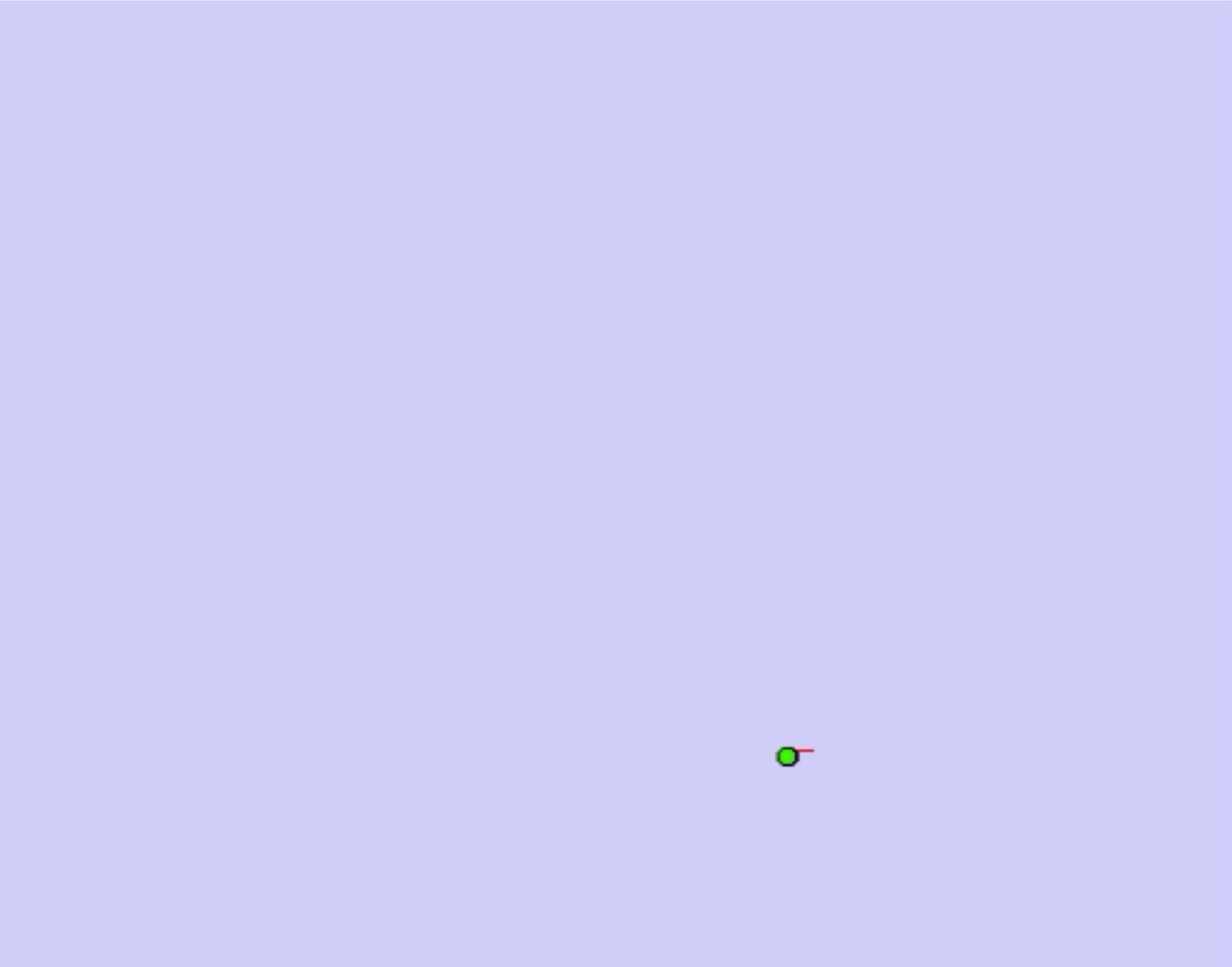


Scan Matching: compare to sensor data from previous scan



Courtesy of S. Thrun

FastSLAM: Particle-Filter SLAM



Courtesy of S. Thrun

Conclusion (thus far ...)

- The presented approach is a highly efficient algorithm for SLAM combining ideas of scan matching and FastSLAM
- Scan matching is used to transform sequences of laser measurements into odometry measurements
- This version of grid-based FastSLAM can handle larger environments than before in “real time”

What’s Next?

- Further reduce the number of particles
- Improved proposals will lead to more accurate maps
- Use the properties of our sensor when drawing the next generation of particles